A
TREATISE
ON
THE DIFFERENTIAL CALCULUS,
AND
ITS APPLICATIONS TO ALGEBRA AND GEOMETRY:
FOUNDED ON THE METHOD OF INFINITESIMALS.

BY

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"Les progrès de la science ne sont vraiment fructueux, que quand ils amènent aussi le progrès des Traités élémentaires."—CH. DUPIN.

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PREFACE.

It was intended that the present volume should be a second edition of "a Treatise on the Differential Calculus founded chiefly on the method of Infinitesimals," which was published in London in the year 1848: but during the process of revision, the matter, as well expository as illustrative, has received such modifications and additions that the Treatise must be considered new.

The character of it is elementary; in it are explained the language, symbols, and first processes of Infinitesimal Calculus; with the view too of exhibiting the principles in a form less repulsive than is usual with English writers, many illustrations are introduced which may appear foreign to the subject; yet they are not so; and in them the thoughtful reader will see, that although he may be ignorant of the nomenclature of the science, yet that neither the idea nor the apparatus is entirely new to him: he will have contemplated them beforehand in their applied and less technical forms. Many examples also are inserted, especially in the early part of the volume, to give the student an aptness in applying general rules to particular cases, and also to present principles in a less abstract form. But moreover, as

the Treatise is intended for students in the universities, it includes the advanced parts of the science, and those theorems which are close upon the boundaries of our knowledge.

The matter was first delivered orally in lecture, and subsequently written; whence has arisen the colloquial style, and which it has been thought expedient to retain, under the conviction that it invests a book with a personal and living character more akin to the explanations of a speaking teacher, and thereby infuses life into what might be otherwise dry text; and in some few passages wherein disputed questions are discussed, objections are stated as if urged by an opponent.

As to the matter apart from the language, the Treatise lays no claim to originality; had it such pretensions, and were they well founded, it might be ill-suited to its didactic and educational object. The sources whence it is derived are various, and for the most part foreign; all cannot be specified, and probably I am indebted to others for more than I am aware of; every book, and perhaps every sentence and every conversation, may leave its impress which, unconsciously to himself, modifies an author's opinions. Almost all which has been taken from other sources, and is not specially acknowledged, has been so long known and commented on, as to have become *publici juris*; a few names occur in the foot- and other notes. I am however under especial obligation to M. Cauchy in his various treatises and memoirs, to his Redacteur M. l'Abbé Moigno, to M. Navier, to the late Mr. D. F. Gregory, to Professor De Morgan, the author of the treatise published by the Society

for the Diffusion of Useful Knowledge, to Professor Donkin and to Mr. W. Spottiswoode, M.A., both of the University of Oxford, and from whom I have received valuable assistance and advice in many parts of the Treatise.

The process of assimilating a body of matter so large, and of such diverse origin, as that of Infinitesimal Calculus, is necessarily long; and in the course of it the question arises, under what principle is all to be harmonized? the parts are seemingly dissonant, what renders them consistent? "whence is the string obtained on which the pearls are to be hung?"

An inquiry of this kind is far too large to be answered within the limits of a preface, but some few remarks are necessary for a due understanding of our method.

Infinitesimal Calculus, both in its pure and applied forms, whether of geometry or of mechanics, is a branch of the science of Number; its symbols are of the same kind, and are operated on according to the same laws; they are applied subject to the same conditions, are interpreted on the same principle, and lead to analogous results. What then is its specific characteristic? In the parts of the science of Number which are supposed to have been previously studied, viz. in arithmetic and algebra (as it is called), numbers are finite and discontinuous; but Number also admits of being continuous, that is, is capable of gradual growth and of infinitesimal increase. Number under this aspect is what Infinitesimal Calculus contemplates: and investigates the new properties of it, the new symbols required to express them, and the

new laws to which they are subject; it has thus to create its own materials, and these materials are infinitesimals.

The method therefore has been at first to unfold those properties of Number which are necessary to the construction of the science of infinitesimals; and then so to describe *continuous* Number and the infinitesimal elements of its growth, in their essential qualities, as to be able to enunciate certain axiomatic properties of them, from which the Calculus may be evolved. These are stated in seven Theorems in Art. 9; of which perhaps the most important is Theorem VI, presenting the character of infinitesimals in their broadest view, viz. that they are such that a finite number of them has no value at all when added to a finite quantity. These Theorems are the ultimate propositions of the science: from them the other truths are inferred, and on them does the correctness of the processes depend. In order to the subsequent use of the advantage in the way of inference possessed by algebraical symbols, we express them in mathematical language, and then proceed to deduce the consequences with which they are pregnant; and this deductive process is continued through the Treatise. In the course of it however, and especially in the applied parts, the subject-matter becomes enlarged, new relations are introduced, and new names are required; thus definitions become necessary, and these are so constructed as to contain in germ the substance of the Articles dependent on them.

A due understanding of these axioms and definitions is plainly of the utmost importance, for on it

does it depend whether we work with mere symbols, or whether the symbols are σημεῖα of philosophical ideas which we comprehend. With the object of guarding against such superficial knowledge, which is useful neither in its results nor as an intellectual exercise, geometrical interpretation of infinitesimals has been often introduced, and magnified diagrams exhibit lines, areas, angles, &c. which are represented by symbols of infinitesimals: every process, nay every step in every process, admits of such geometrical translation; and it is most desirable that the student should exercise himself in it: by so doing he will have a most certain test whether his operations are according to the laws of correct inference, or whether he is merely applying mnemonic rules and groping his way in the dark by some obscure road, and drawing his conclusion as it were only by riddles.

The Calculus then is that of infinitesimals; but as the most convenient mode of forming them shews them in the light of differences between two states at a "very small" distance apart, they have obtained the name of Differentials; and hence the name of "Differential Calculus" has arisen; the latter term is retained, although it refers to the mode of generating the materials, and not to any pregnant property of them when generated. The notation also and language of Lagrange's Calculus of derived-functions has been employed, because it expresses in a most convenient form some of the earliest results of operations with infinitesimals, and their relations to finite quantities: but I would warn the reader especially against supposing that our Calculus is founded on the notion of derived-functions as the

coefficients of the terms of a series; they are not considered in that relation at their first admission, and it is only by a course of reasoning that they afterwards become so. Expansion in a series has been admitted fundamentally but once, viz. in Art. 21, and whatever may be the final issue of the question as to Convergent and Divergent Series, it cannot affect that which I have employed, inasmuch as it is proved to be Convergent.

The Volume consists of two Parts; in the former of which are investigated the Theorems of the Differential Calculus so called, and in the latter the applications to Geometry of two and three dimensions are discussed.

The Chapters mark the salient divisions of the matter; and these again are sub-divided into Articles, which for the sake of reference are numbered continuously through the Volume, and have their numerals placed in the corners on the top of the pages. The bracketed numerals attached to the more important equations are separate for each Chapter; and the references are usually made to the numbers of the equation and of the Article. The Analytical Table of Contents exhibits a résumé of the matter of the Treatise.

PEMBROKE COLLEGE, OXFORD,
Sept. 3, 1852.

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CORRECTIONS.

Page 22, line 24, for "of" read "is"

22, — 29, for "$1+x$" read "$1-x$"

23, — 27, for "IV" read "VI."

31, — 10 for "Differentation" read "Differentiation"

49, — 15, for "y" read "Δy"

52, — 1, for "b" read "$-bdx$"

53, — 28, for "(x^2-2)" read "$(x^2-2)^2$"

91, — 3, for "$n+\frac{\pi}{2}$" read "$x+\frac{\pi}{2}$"

98, — 6, for "$z+\frac{1}{z}$" read "$z-\frac{1}{z}$"

101, — 24, for "$\frac{n-2}{2}\pi$" read "$\frac{n-2}{n}\pi$"

115, — 12, for "the equicrescent" read "equicrescent"

123, — 2, for "$du\,dx$" read "$dz\,dx$"

124, — 22, for "$d\mathrm{F}'$" read "$d\mathrm{F}$" twice.

171, — 10, for "$x_n-x_0=h$" read "$x_n=x_0+h$"

180, — 8, for "$(\sin i)^2$" read "$(\sec i)^2$"

—, — 9, for "$4(\sec i)^2$" read "$-4(\sec i)^2$"

247, — 12, for "dx" read "dy"

250, — 23, for "a_2" read "a_3"

277, — 1, fot "MO^2" read "MQ^2"

288, — 8, omit "185.]"

—, — 16, for "186.]" read "185.]"

317, — 12, for "series" read "species"

336, — 21, for "series" read "species"

343, — 20, for "π" read "2π"

INFINITESIMAL CALCULUS.

PRELIMINARY THEOREMS.

As the following Algebraical Theorems will be frequently applied in the course of the treatise, it is convenient to prove them once for all, and, for the sake of reference, to place them at the beginning of the work.

I.

If there be any number of *equal* fractions $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \dots \frac{a_n}{b_n}$, each of them is equal to

$$\frac{a_1+a_2+a_3+ \dots\dots +a_n}{b_1+b_2+b_3+ \dots\dots +b_n};$$

and, if $m_1\ m_2\ m_3 \dots\dots m_4$ be any multipliers, to

$$\frac{m_1\,a_1+m_2\,a_2+m_3\,a_3+ \dots\dots +m_n\,a_n}{m_1\,b_1+m_2\,b_2+m_3\,b_3+ \dots\dots +m_n\,b_n};$$

and to

$$\frac{\{a_1^2+a_2^2+a_3^2+ \dots\dots +a_n^2\}^{\frac{1}{2}}}{\{b_1^2+b_2^2+b_3^2+ \dots\dots +b_n^2\}^{\frac{1}{2}}}.$$

Let each of the fractions $= r$, that is, let

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \dots\dots = \frac{a_n}{b_n} = r;$$

therefore

$$\begin{array}{lll} a_1 = b_1\,r & m_1\,a_1 = m_1\,b_1\,r & a_1^2 = b_1^2\,r^2 \\ a_2 = b_2\,r & m_2\,a_2 = m_2\,b_2\,r & a_2^2 = b_2^2\,r^2 \\ \dots & \dots & \dots \\ a_n = b_n\,r & m_n\,a_n = m_n\,b_n\,r & a_n^2 = b_n^2\,r^2. \end{array}$$

Whence, by addition and division,

$$r = \frac{a_1 + a_2 + a_3 + \ldots\ldots + a_n}{b_1 + b_2 + b_3 + \ldots\ldots + b_n};$$

$$= \frac{m_1\, a_1 + m_2\, a_2 + m_3\, a_3 + \ldots\ldots + m_n\, a_n}{m_1\, b_1 + m_2\, b_2 + m_3\, b_3 + \ldots\ldots + m_n\, b_n};$$

$$= \frac{\{a_1^2 + a_2^2 + a_3^2 + \ldots\ldots + a_n^2\}^{\frac{1}{2}}}{\{b_1^2 + b_2^2 + b_3^2 + \ldots\ldots + b_n^2\}^{\frac{1}{2}}} = \frac{a_1}{b_1} = \frac{a_2}{b_2} = \ldots\ldots = \frac{a_n}{b_n}.$$

Q. E. D.

The geometrical form of the first of these theorems is the twelfth proposition of the fifth book of Euclid.

II.

When three unknown quantities are involved in three (so called) *linear* equations of the forms

$$a_1x + b_1y + c_1z = d_1 \qquad (1)$$
$$a_2x + b_2y + c_2z = d_2 \qquad (2)$$
$$a_3x + b_3y + c_3z = d_3 \qquad (3)$$

the following is the most convenient method of determining each of the unknown quantities in terms of the constants.

Multiply (1) by λ_1, (2) by λ_2, (3) by λ_3, and add; then

$$(a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3)\, x + (b_1\lambda_1 + b_2\lambda_2 + b_3\lambda_3)\, y$$
$$+ (c_1\lambda_1 + c_2\lambda_2 + c_3\lambda_3)\, z = d_1\lambda_1 + d_2\lambda_2 + d_3\lambda_3.$$

As three undetermined quantities, viz. λ_1 λ_2 λ_3, have been introduced, we may make three suppositions respecting them: leaving one to be determined hereafter, let two be, that the coefficients of y and z be equal to zero; so that

$$b_1\lambda_1 + b_2\lambda_2 + b_3\lambda_3 = 0$$
$$c_1\lambda_1 + c_2\lambda_2 + c_3\lambda_3 = 0;$$

whence by elimination,

$$\frac{\lambda_1}{b_2c_3 - c_2b_3} = \frac{\lambda_2}{b_3c_1 - c_3b_1} = \frac{\lambda_3}{b_1c_2 - c_1b_2}.$$

But thus the ratio only of the multipliers has been determined, and therefore *any* numbers bearing to one another the above ratio would satisfy the requisite conditions; to take however the most simple numbers, let the third condition be introduced,

and be, that each of the above fractions be equal to unity; and therefore

$$\lambda_1 = b_2c_3 - c_2b_3$$
$$\lambda_2 = b_3c_1 - c_3b_1$$
$$\lambda_3 = b_1c_2 - c_1b_2;$$

and therefore

$$x = \frac{d_1(b_2c_3-c_2b_3)+d_2(b_3c_1-c_3b_1)+d_3(b_1c_2-c_1b_2)}{a_1(b_2c_3-c_2b_3)+a_2(b_3c_1-c_3b_1)+a_3(b_1c_2-c_1b_2)}.$$

By similar processes, if the coefficients of z and x had been equated to zero, the result would have been

$$y = \frac{d_1(c_2a_3-a_2c_3)+d_2(c_3a_1-a_3c_1)+d_3(c_1a_2-a_1c_2)}{b_1(c_2a_3-a_2c_3)+b_2(c_3a_1-a_3c_1)+b_3(c_1a_2-a_1c_2)},$$

and if the coefficients of x and y had been equated to zero,

$$z = \frac{d_1(a_2b_3-b_2a_3)+d_2(a_3b_1-b_3a_1)+d_3(a_1b_2-b_1a_2)}{c_1(a_2b_3-b_2a_3)+c_2(a_3b_1-b_3a_1)+c_3(a_1b_2-b_1a_2)}.$$

This method of elimination is generally known by the name of Lagrange's Rule of Cross-Multiplication, the origin of which term is sufficiently obvious from the *form* of the multipliers; two examples are subjoined for the sake of practice, but the student is recommended to exercise himself in many others.

Ex. 1.

	x	y	z
$2x+4y+5z = 49$	2	12	−11
$3x+5y+6z = 64$	−1	−12	10
$4x+3y+4z = 55$	−1	3	−2

the values of the multipliers corresponding to the several variables being arranged in the annexed vertical rows: whence

$$\left.\begin{array}{lll} x\,(\ \ 4-\ 3-4) = & 98-\ 64-\ 55 & \therefore\ x=7 \\ y\,(\ \ 48-60+9) = & 588-768+165 & \ y=5 \\ z\,(-55+60-8) = & -539+640-110 & \ z=3 \end{array}\right\}$$

Ex. 2.

	x	y	z
$3x-7y+4z = 1$	7	4	1
$-5x+9y-z = 22$	−1	−1	−1
$x-2y+z = 0$	−29	−17	−8

the values of the several multipliers being arranged as before:

whence

$$\left.\begin{array}{l} x\,(\ \ 21+5-29) = 7-22 \ \ \therefore\ \ x = 5 \\ y\,(-28-9+34) = 4-22 \ \ \therefore\ \ y = 6 \\ z\,(\ \ \ 4+1-\ 8) = 1-22 \ \ \therefore\ \ z = 7 \end{array}\right\}$$

III.

If α_1, α_2, α_3, ... α_n are quantities of the same sign, and a_1, a_2, a_3, ... a_n be any other homogeneous quantities capable of addition and subtraction, then $a_1\ \alpha_1 + a_2\ \alpha_2 + a_3\ \alpha_3 + \ldots\ldots + a_n\ \alpha_n$ is equal to $(\alpha_1 + \alpha_2 + \alpha_3 + \ldots\ldots + \alpha_n)$ multiplied into some quantity greater than the least, and less than the greatest, of the quantities a_1, a_2, a_3, ... a_n.

The proof of this proposition depends on the fact, if both terms of an inequality are multiplied or divided by a positive number, the sign of inequality remains the same; that is, the quantity which was greater before the multiplication is the greater after it; but if the terms are multiplied by a negative number, the sign of the inequality is reversed: that is, a $>$ is changed into a $<$, and a $<$ into a $>$. This is easily shewn by an example; as, for instance, 5 is greater than 2; let each side be multiplied by $+4$, then $20 > 8$; again, let each side be multiplied by a negative number, as -2: the $>$ is changed into a $<$, $-10 < -4$, because -10 is less than -4.

Let L be the least and G the greatest of the quantities a_1, a_2, a_3, ... a_n; then, with the exception of the two cases of the greatest and the least of the quantities, whereby however the final result is not vitiated, we have the following inequalities:

$$\begin{array}{l} a_1 \text{ is } > \text{L}, < \text{G} \\ a_2 \text{ is } > \text{L}, < \text{G} \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ a_n \text{ is } > \text{L}, < \text{G}; \end{array}$$

First, let the quantities α_1, α_2, α_3, ... α_n be positive, so that the signs of the above inequalities will not be changed, when they are multiplied as follows:

$$\alpha_1 a_1 \text{ is } > \text{L} a_1, < \text{G} a_1$$

$$\alpha_2 a_2 \text{ is } > \text{L} a_2, < \text{G} a_2$$

$$\cdots\cdots\cdots$$

$$\alpha_n a_n \text{ is } > \text{L} a_n, < \text{G} a_n;$$

and therefore by addition

$$\alpha_1 a_1 + \alpha_2 a_2 + \ldots\ldots + \alpha_n a_n \text{ is } > \text{L}\ (a_1 + a_2 + \ldots\ldots + a_n)$$
$$< \text{G}\ (a_1 + a_2 + \ldots\ldots + a_n)$$

$$\therefore\ \alpha_1 a_1 + \alpha_2 a_2 + \ldots\ldots + \alpha_n a_n = (a_1 + a_2 + a_3 + \ldots\ldots + a_n) \times \text{some } \textit{mean} \text{ value of the } \alpha\text{s},$$

signifying by *mean* value, a quantity > than the least, and < than the greatest.

Secondly, if a_1, $a_2 \ldots\ldots a_n$ be negative, the signs of the inequalities would have been changed in the first multiplication, and would have been again changed in the final result, because $a_1 + a_2 + \ldots\ldots + a_n$ would be a negative quantity, and thus the same result would follow: and therefore the proposition is proved.

IV.

If $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots\ldots \frac{a_n}{b_n}$ be a series of fractions, the numerators of which are of either sign, and *the denominators all of the same sign*, then $\frac{a_1 + a_2 + \ldots\ldots + a_n}{b_1 + b_2 + \ldots\ldots + b_n}$ is equal to some quantity greater than the least and less than the greatest of the given fractions.

First, let all the denominators be positive, and let L be the least and G the greatest of the fractions; then

$$\frac{a_1}{b_1} \text{ is } > \text{L}, < \text{G}$$

$$\frac{a_2}{b_2} \text{ is } > \text{L}, < \text{G}$$

$$\cdots\cdots$$

$$\frac{a_n}{b_n} \text{ is } > \text{L}, < \text{G}.$$

Let the inequalities be severally multiplied by the positive quantities $b_1, b_2, b_3 \ldots\ldots b_n$, by which process the signs are not changed; then

$$a_1 \text{ is } > \mathrm{L}b_1, < \mathrm{G}b_1$$

$$a_2 \text{ is } > \mathrm{L}b_2, < \mathrm{G}b_2$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$a_n \text{ is } > \mathrm{L}b_n, < \mathrm{G}b_n,$$

and therefore by addition

$$a_1+a_2+a_3+ \ldots\ldots +a_n \text{ is } > \mathrm{L}\,(b_1+b_2+b_3+ \ldots\ldots +b_n)$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \text{ is } < \mathrm{G}\,(b_1+b_2+b_3+ \ldots\ldots +b_n),$$

and therefore

$$\frac{a_1+a_2+a_3+ \ldots\ldots +a_n}{b_1+b_2+b_3+ \ldots\ldots +b_n} \text{ is } > \mathrm{L}, < \mathrm{G}, \text{ and therefore is equal}$$

to some *mean* value of the fractions.—Q. E. D.

Secondly, let $b_1, b_2, b_3 \ldots\ldots b_n$ be negative; then, as before,

$$\frac{a_1}{b_1} \text{ is } > \mathrm{L}, < \mathrm{G}$$

$$\frac{a_2}{b_2} \text{ is } > \mathrm{L}, < \mathrm{G}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\frac{a_n}{b_n} \text{ is } > \mathrm{L}, < \mathrm{G}.$$

Let these inequalities be severally multiplied by the negative quantities $b_1, b_2, b_3 \ldots\ldots b_n$, so that the signs of them are changed; then

$$a_1 \text{ is } < \mathrm{L}b_1, > \mathrm{G}b_1$$

$$a_2 \text{ is } < \mathrm{L}b_2, > \mathrm{G}b_2$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$a_n \text{ is } < \mathrm{L}b_n, > \mathrm{G}b_n;$$

$$\therefore \quad a_1+a_2+a_3+ \ldots\ldots +a_n \text{ is } < \mathrm{L}\,(b_1+b_2+ \ldots\ldots +b_n)$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \text{ is } > \mathrm{G}\,(b_1+b_2+ \ldots\ldots +b_n);$$

whence, bearing in mind that the sign of inequality is changed by dividing by a negative quantity,

$$\frac{a_1+a_2+a_3+ \ldots\ldots +a_n}{b_1+b_2+b_3+ \ldots\ldots +b_n} \text{ is } > \mathrm{L}, < \mathrm{G}.$$

Q. E. D.

DIFFERENTIAL CALCULUS.

PART I.

ANALYTICAL INVESTIGATIONS.

CHAPTER I.

GENERAL PRINCIPLES, AND EXPLANATION OF TERMS.

ARTICLE 1.] Infinitesimal Calculus is a branch of that science, the aggregate of the rules and operations of which French writers call "Le Calcul," but for which we have no more specific name than the Science of Number*; a metaphysical inquiry into what *number* is, and whence it springs, would be out of place in a didactic treatise such as the present, and it will be sufficient for the student to have that notion of it which an ordinary knowledge of arithmetic and algebra implies; but it is well to recal his attention to certain axiomatic properties of it, and to bring into greater prominence those from which the Infinitesimal Calculus is deduced.

Number is *the ratio or relation which two quantities of the same kind bear to one another in respect of quantuplicity*. By quantity I mean whatever is capable of measurement; whether it be geometrical space, or weight, or time, or heat, or light, or velocity, or any thing else; that, viz., of which we can predicate muchness in reply to the question "how much?" or number of times in answer to "how many times?" The above phænomena may be severally the substrata of mixed sciences, as they are called, but they can only be treated of in accordance with the rules of the science of number, because they are

* M. Comte writes, "Le Calcul a pour objet propre de résoudre toutes les questions de *nombres*."—*Philosophie Positive*, vol. i. p. 143.

capable of addition and division and measurement, at least in conception, if not in act; and it is only when a problem can be reduced to a question of pure number that it can be brought within the domain of the Calculus; and when this condition is satisfied, the science of number supplies, as it were, the skeleton or framework on which such mixed science is treated; and considers the subject-matter not in its concrete and physical and phænomenal, but in its abstract state, as measured, and in the measure, as the correct representative of it. It does not take cognizance of this or that weight or colour, but of the *number of times* such a weight contains another weight, and so on; and this is what I mean by the terms "in respect of quantuplicity" in the above definition of number; and thus it is that the science is so general, almost universal, because all the subject-matter of the so called Physical Sciences conforms to its requirements in respect of admitting of measurement.

2.] There are two modes of measuring quantities, and thus of arriving at abstract numbers from the concrete magnitudes. Firstly, a certain amount of the given quantity is taken, which for the sake of convenience is called an unit of that particular quantity, and with it any other amount is compared, the principle of comparison being assigned by the particular science whose subject-matter the quantity is; and if the latter amount be divisible into two, or three, or more parts, severally equal to each other and to the unit, we say that the latter amount is twice, or three times, or more times the unit, and thus arrive at the abstract number; and the problem of determining other relations arising out of this one belongs to the science of number. The unit, it is to be observed, is arbitrary; thus, for instance, if a certain volume of matter of given density, say a cubic inch of distilled water, be considered the volume-unit, and its density be called the density-unit, then if two cubic inches of distilled water could be compressed into one inch, its density would be *two*. Hereby we arrive at abstract number by directly comparing any given concrete quantity with the unit of that quantity. Secondly, we may estimate quantity without *formally* introducing the unit; for suppose of two unequal quantities, one to be divisible into two equal parts, and the other into three parts, equal to each other and to each of the divided parts of the former quantity; then, although if one

of the equal parts were taken as the unit, one quantity would contain two and the other three units: yet we may omit the unit, and say that the ratio of one quantity to the other is that of the numbers two to three. Thus, if two lines admit of being resolved into 3 and 7 parts respectively, which are equal to each other, the ratio of the lines will be represented by the numerical ratio 3 : 7; and this mode of measuring quantities is independent of the amount of the chosen unit; for if the above lines had been divided into 6 and 14 equal parts respectively, or into $3n$ and $7n$ equal parts respectively, the unit would have been only $\frac{1}{2}$ or $\frac{1}{n}$th part of what it was in the previous resolution; and yet the ratio of the lines, or of the numbers which represent them, viz. 6 : 14, or $3n : 7n$, would have been the same as before. And this mode of representing lines by numbers is equally applicable to areas, hours, weights, &c. all of which may be so related to each other as to admit of resolution into 3 and 7 equal parts respectively, and thus be represented by the ratio of the numbers 3 : 7. It is manifest that only quantities of the same kind can be measured in either of the above methods, and hence it is only from comparing *homogeneous* quantities that numbers can be formed. The principle and mode of comparison however must be assigned by the particular science whose subject-matter the quantity is; thereby are its concrete materials abstracted and brought within the range of the Science of Number, which has thus to deal with only abstract quantuplicities; and thus (which is a point of the utmost importance, and deserves the most careful attention) the symbols 2, 3, 4, ... $a, b, c,$ ...... $x, y, z,$ do not represent concrete quantities, such as 2 *ounces*, or a *hours*, or x *feet*, but *abstract numbers*; and it is the properties of these that the Science of Number has to discuss*.

3.] The distinguishing characteristic of such numbers is, that they remain after any operation the same in *kind* that they

* The two modes of measuring quantities and of thereby forming numbers, correspond to the two aspects in which arithmetical fractions are viewed; one in which the denominator assigns the unit, and the numerator the number of times it is to be taken, and the other in which they are considered as the expressions of arithmetical ratios. See Peacock's Algebra, 2nd edition, vol. i. pp. 54, 155.

were before. When two numbers are added or subtracted, the sum or difference is a number; when two numbers are multiplied or divided, the resultant is the same in kind as each of the components: as they are numbers, so is it number. Thus $2 \times 3 = 6$, and 6 is an abstract number of the same kind as 2 and 3; that is, twice thrice is equivalent to six times; $8 \div 4 = 2$, that is, one-fourth part of eight times is twice. The same is also true of the operations of Involution and Evolution. These remarks are important, because if the symbols represent concrete quantities the results would be otherwise; thus in the *analogous* operation of geometrical multiplication, if two linear inches be "multiplied" by three linear inches, the result is six square inches: and therefore in the process we have changed from linear to superficial quantity. And so, if the six superficial inches be multiplied by four linear inches, the result is 24 cubic inches, or inches of volume, and thus by the last process we have passed from superficial magnitude to solid content. By the operations then the kind has changed; and as all the dimensions which space admits of have been exhausted, it is impossible to multiply together more than three geometrical lines; thus two linear inches multiplied into themselves four times is an impossibility; or, in other words, a geometrical line cannot be raised to any power above the third. And the possibility of extracting roots is confined within narrower limits; thus the square root of a superficial area is a possible quantity, and so is the cube root of a solid content, being in each case a line. The case of geometrical multiplication is the most favourable one, for in other subject-matter, which is only uni-dimensional, we cannot at all multiply the concrete quantities, and can only divide them when they are homogeneous. It is absurd to speak of a pound multiplied into a pound, or of the product of an hour by an hour: such operations are impossible, and have no meaning; but we can divide two pounds by one pound, and thereby arrive at the abstract number 2, because the inverse process is possible, and may therefore be undone: that is, because we can multiply any concrete unit by an abstract number; but we can no more divide pounds weight by pounds sterling than we can multiply hours by degrees of heat. These remarks on the abstract character of Number are relevant to

the present subject, inasmuch as they prove the correctness of such an expression as

$$ay^2 + bxy + cx^2 + dy + ex + f,$$

for each symbol by itself expressing a number, each term, whether consisting of one or two or three factors, is a number also: and thus they may be added, and the whole expression is homogeneous and correct. Whereas, did such symbols represent concrete quantities, as, for example, geometrical lines, the first three terms would represent solid content, the next two superficial area, and the last lineal length: and thus they would be heterogeneous, and could not be added. Also in such a point of view a term ay^3, consisting of more than three dimensions, would be uninterpretable and impossible.

Hence we conclude, that

(1.) The symbols whose laws and combinations are considered in the Science of Number express the number of times any thing is taken; and the science, disregarding the concrete thing, discusses the properties of the abstract number.

(2.) As Infinitesimal Calculus is a branch of the Science of Number, the symbols which will be employed in the following work represent number only; and the properties of number will be considered only as they are represented by symbols.

This latter conclusion is important, as it restricts the subject-matter to symbols, and our discussion to their laws and properties.

4.] The numbers or quantities which are employed in the following treatise are of two kinds, *constants* and *variables*.

Constant numbers are those which have the same determinate value throughout a given operation or problem: though in another operation, or considered in another relation, they may vary. Such are the symbols 2, 3, 4, and for the most parts those in algebra, which are represented by the first letters of the alphabet: constant numbers are specific in form and value.

A *variable* number is that which is capable of receiving values different from each other, and generally admits of any value, though it may by the conditions of a problem be restricted to values of a particular kind, or within certain limits; variable quantities are *general* in form, though they admit of specific

values: they are generally represented by the last letters of the alphabet.

5.] In order to avoid misconception as to the following terms, which will be frequently employed, it is necessary to give detailed explanations of them; viz. *Definite*, *Indefinite*; *Infinite*, *Finite*, *Infinitesimal*.

Definite means *determinate* and *assigned*; thus constants are definite. Indefinite numbers are those whose values are not assigned; they are represented by general symbols, and may therefore, so far, have any value.

To form an accurate and due conception of the latter three terms, and of the means of symbolizing and estimating them, requires a knowledge of the whole Calculus: we must therefore have recourse to analogous illustration, in hope that the student may glean from it such notions, imperfect though they will be, as will enable him to understand the technical language of the science.

By *finite* we generally mean that which is within reach, or may be brought within reach, of our senses. Thus a ton, or an ounce, may be taken as the unit of weight, and any number of tons or ounces which the senses perceive would be considered a finite weight; and many animalculæ, which, on account of their minuteness, are beyond the power of unassisted vision, would nevertheless be considered finite, because they may be brought within it by means of the microscope, and may be measured. Or again, inasmuch as the senses are the media by which impressions of external objects are conveyed to the mind, and as the mind conceives them when so conveyed, we apply the term *finite* to those magnitudes, the relation of which to other magnitudes of the same kind the mind is capable of conceiving*. The powers therefore of our senses and mind place the limit to the finite; but those magnitudes which severally transcend these limits by reason of their being too great or too small, we call *infinite* and *infinitesimal* (or infinitely small). Thus, when our senses fail to perform their office of transmitting to the mind what it would think about, by reason of the object being too large or too small, or when for a similar reason the mind fails to be capable of considering the relation of such objects, and when the most delicate subsidiary instru-

* Peacock's Algebra, vol. ii. p. 294.

ments for assisting the senses are employed to bring within their reach what was beyond them, and yet in vain, then we are on the boundary of the infinite or infinitesimal: of the infinite if the object be too vast, of the infinitesimal if it be too minute. Physical Science affords instances of both these cases. The distances of those fixed stars of which the parallax has not been discovered must be so great, that 400 millions of miles are not appreciable in comparison of them; considering then these millions of miles to be a finite quantity, yet no *sensible* change is made in the distance of a star by the addition or subtraction of them. Nay, more than this; we can employ our millions of miles to greater advantage; we can make them the base of a triangle whose vertex is the star, and yet, great as is the delicacy of our astronomical instruments, the sides of the triangle are to all appearance parallel. This then is a case where we cannot compare two geometrical distances, on account of the immensity of one of them. Considering then the 400 millions of miles to be a *finite* quantity, the distance of the star is *infinite*. Again: if one grain weight of aloetic acid be added to five pounds of pure water, the whole will after a short time assume a fine crimson colour, which could not happen unless the grain of aloetic acid had been divided and equally diffused throughout the whole volume. Now it is possible to see a quantity of water as small as a thousandth part of a grain, and such a portion of the solution would contain a thirty-five millionth part of a grain of aloetic acid. We have therefore actually divided this substance into thirty-five millions of parts, and the most delicate microscope does not so far magnify the atoms of the acid that they should be separately visible in the water; yet there they are, and are so small as to be beyond the limit of our vision, even though it be increased many thousand times: they are *infinitesimal,* though the sum of them is finite; and as they are so small, there must be an infinity of them. Hence also we have a new aspect of such quantities. In reference to a finite quantity, infinity and infinitesimal are reciprocal terms, each implying the other; the finite quantity may be the infinitesimal infinitely-quantupled, and the infinitesimal an element of the finite quantity, when it is resolved into an infinity of parts. Again: an infinite quantity may be so large, as not only to surpass the compass of our senses, but

also to surpass quantities which are from their magnitude beyond them; that is, there may be infinite quantities beyond infinite quantities, and others again beyond these: and thus there may be quantities infinitely greater than infinities, and there may be *orders of infinities*. Astronomy supplies instances of such quantities, to an extent within the reach of sight, by means of the telescope, but beyond the range of the micrometer as a means of measuring distance; assuming as the unit of length the mean radius of the earth's orbit, which is about 95 millions of miles, we can compare with it the mean radii of the orbits of the other planets, and thus determine the *relative* sizes of their orbits; but when we extend our observations to other bodies in the celestial space, we find stars situated at such a distance from the sun, that, taking the star to be the vertex of an isosceles triangle, and the base to be a line through the sun's centre and of 190 millions of miles in length, the vertical angle of the triangle is less than 1″, and in the case of Capella is computed to be 0″.046; were the vertical angle 1″, it can easily be shewn that the distance would be 20 billions of miles; and as it is determinable, we may say that it is comparable with finite quantities, though on the verge of the infinite. "In such numbers the imagination is lost; the mode we have of conceiving such intervals at all is by the time which it will take light to traverse them. Light, we know, travels at the rate of 192 thousand miles per second; it will occupy therefore three years and eighty-three days to traverse the distance in question. Now as this is an inferior limit, which it is already ascertained that even the brightest and therefore (in the absence of all other indications) the nearest stars exceed, what are we to allow for those innumerable stars of the smaller magnitudes which the telescope discloses to us? What for the dimensions of the galaxy in whose remoter regions the united lustre of myriads of stars is perceptible in powerful telescopes as a feeble nebulous gleam*?" Here then we have not only finite but also infinite distances, and spaces infinitely greater than these infinite distances; that is, we have successive orders of infinities, and in an ascending scale from finite distances, of which our senses are cognizant, to those infinite spaces which surpass our powers of measurement.

* Sir John Herschel's Outlines of Astronomy, art. 800 and following.

So again may any one of the small particles of the aloetic acid which has been dissolved in the water be conceived to be analysed into other parts infinite in number, each one of which will therefore be infinitesimally small in comparison of its original particle; that is, one small particle may be conceived to be distributed through a finite volume, and thus to be resolved into other particles infinitely less than itself: and thus we may arrive at orders of infinitesimals, each one being infinitesimally less than that of which it is an element. It is also to be observed, that in the resolution of a finite quantity into infinitesimals of successive orders, different orders of infinities arise which severally correspond to the orders of infinitesimals, to which they are so related that the product of the infinity and infinitesimal of the same order is equal to the original finite quantity.

Here perhaps it may be asked, *when* does a quantity pass from the finite to the infinite and to the infinitesimal? How many finite quantities must be added to make an infinity, and into how many parts must a finite quantity be resolved so that each should be infinitesimal? An answer to such questions may be beyond our power; and it may be a matter of words only: but it is also beside the object and requirements of the Calculus. We have nothing to do with concrete quantities; the instances above cited are for the sake of *illustration* only: to give the reader a rough notion of the principles; such as may serve their purpose until other and more accurate ones take their place; our subject-matter is *number*, and number as represented by symbols: and the form of the symbols, and the subsidiary symbols which will be derived from them, as will be shewn in the sequel, enable us to overcome the apparent difficulty.

6.] To resume then the course of the exposition from the end of the 4th article: variable number may change value in two ways, either *continuously* or *discontinuously*.

A *quantity* or *number* varies discontinuously when it passes abruptly from one value to another, as by the addition of a finite quantity. Thus the passage from 1 to 2, and from 2 to 3, and so on, is made discontinuously, viz. by the abrupt addition of the number 1; similarly we pass from $2x$ to $4x$, and from $4x$ to $8x$, by the successive addition of $2x$ and $4x$; thus the changes are made "per saltus."

But continuous increase is when number *grows*, that is, passes from one value to another only by going through all the intermediate numbers, whereby the successive increments or augments which the numbers receive are infinitesimal; thus, if we pass from 3 to 4 not only by going through 3.1, 3.2, 3.3, and if we pass from 3 to 3.1 not only by going through 3.01, 3.02, 3.03, ... and if we pass from 3 to 3.01 not only by going through 3.001, 3.002, ... and so on to any *finite* number of divisions, the increase is discontinuous; but if the number of divisions be infinite, and if the lesser number pass into the greater number by receiving at each successive step an infinitesimal increase, the mode of increase is continuous. For the sake of illustration let us consider the case of motion. Consider the gliding motion of a worm, and suppose it to pass uniformly over an inch in a minute; if the space through which the worm has passed be estimated at the end of each minute only, the space will apparently be discontinuously increased by an inch: but if the space be measured at the end of each infinitesimal or very short lapse of time, the increase during that instant will be very small, and if the instants be infinitesimal, the space, we say, will have increased by infinitesimal increments. The earth's motion in its orbit, the running out of water, the gradual radiation of heat, the growth of a tree, are all instances of a similar continuous increase; and it is worth observing, that it was from such cases that the Calculus had its origin. But if we count horses or men, we count discontinuously: we pass "per saltum" from one man to two men; we cannot divide a man into infinitesimal elements; each man is an unit whose personal existence does not admit of such infinitesimal subdivision. Hence it appears, that numerical continuity requires infinite numerical divisibility, and expresses the property of quantity considered under the aspect of generation by growth: thus the difference of the two modes of increase is one of degree and not of kind. Hence we have a criterion of them; the difference between two successive numbers is finite or infinitesimal, according as the mode of increase is discontinuous or continuous.

The subject-matter of arithmetic and of algebra (commonly so called) is discontinuous number. The numbers 8, 9, 10, ... $a, b, c, \ldots\ldots x, y, z, \ldots\ldots$ as they are commonly employed, are

discontinuous; we pass from one to another "per saltum," and do not contemplate the mode of continuous increase. The distinction between the two sciences appears to be the following: In arithmetic are discussed the properties of numbers which have certain determinate values, and can have none other; in algebra we treat of symbols which are general in form, and either have specific values (as the constants a, b, c), or admit of having one or more such values (as the variables x, y, z).

7.] Infinitesimal Calculus considers number in its aspect of continuous growth. In this lies its distinctive character: it is thus completory to the other two branches of the science; for whereas they treat of finite and discontinuous number, it treats of continuous, and especially of infinite and infinitesimal number; and however vague these terms may appear at first, yet as we shall have to treat of them in their symbolized state, our final results will not be wanting in precision.

The value towards which an expression converges nearer than by any assignable difference, while the symbol on which it depends approaches to any assigned value, is called a *limit* or *limiting value*. If the assigned value of the symbol be zero, the limit is called the *inferior limit*; and if the value be infinity, it is called the *superior limit*.

Thus the inferior limit of $\frac{1}{1+x}$ is 1; although for every value of x greater than 0 the quantity is less than 1, yet the nearer x approaches to 0, the less becomes the difference between $\frac{1}{1+x}$ and 1; and the superior limit is 0; for as x increases, the quantity becomes less and less, and ultimately, when x is greater than any assignable quantity, the difference between $\frac{1}{1+x}$ and 0 is less than any quantity, and thus the limit is zero. So again, as the difference between x and -1 becomes less than any assignable quantity, $\frac{1}{1+x}$ approaches to infinity; similarly the inferior limit of $\tan x$ is 0, and, as x becomes $\frac{\pi}{2}$, the differ-

ence between tan $\frac{\pi}{2}$ and infinity vanishes, and infinity is the limit of tan $\frac{\pi}{2}$.

Again: suppose that we have a series of the form

$$\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \ldots\ldots \frac{n+1}{n},$$

the number of terms of which is infinite; although the terms become less and less as we proceed from left to right, yet the value of the last term does not approach to 0 but to unity, as is manifest from the form of it; unity then is the limit of the last term of the series.

In geometry a circle is the limit towards which the perimeter of an inscribed polygon converges, as the number of sides is infinitely increased, and as thereby the lengths of the sides become infinitesimally small. Many examples of finding limits will be given in the sequel. Hence *zero* is the inferior limit of an infinitesimal, and infinity is the superior limit of a quantity which is greater than any assignable quantity.

The symbols by which we shall represent infinity and an infinitesimal are ∞ and 0: the relation of which is, that if a represent a finite quantity, $\infty = \frac{a}{0}$ and $0 = \frac{a}{\infty}$. We shall attach a more definite meaning to these relations, if we consider a dividend to be the product of the divisor and quotient; thus there is no *finite* quantity which, when multiplied into zero or an infinitesimal, will produce a finite product: nothing short of infinity can do it; and from the illustrations of article 5 it appears that it must be an infinity of a particular kind.

8.] If then any finite numerical quantity be divided into any number of equal or unequal parts, as the case may be, the larger the number of parts is, the smaller is each part; and if the number of parts be infinitely great, each part is an infinitesimal: and the less the difference be between the number of parts and absolute infinity, the less is also the difference between each part and absolute zero. Suppose then a to be a finite determinate quantity, and to be divided into x equal parts; each part $= \frac{a}{x}$; and then, if x be infinitely great, $\frac{a}{x}$ is an infini-

tesimal, x and $\frac{a}{x}$ being thus symbols of an infinity and an infinitesimal, which mutually imply and are reciprocal to each other. Suppose again, $\frac{a}{x}$ to be divided into x equal parts, then each part is equal to $\frac{a}{x^2}$, and x^2 is the number of parts into which a has been divided: thus x^2 and $\frac{a}{x^2}$ severally represent an infinity and an infinitesimal, which are reciprocal to and mutually imply each other. By similar and subsequent divisions we may find x^3, $\frac{a}{x^3}$, ... x^n, $\frac{a}{x^n}$, and therefore other infinities and infinitesimals which are relative to each other.

Or again, suppose i to be an infinitesimal element of a, so that a is divided into $\frac{a}{i}$ equal parts, then $\frac{a}{i}$ is an infinity, and is relative to the infinitesimal i. And again, suppose a to be resolved into elements each of which is equal to i^2, then $\frac{a}{i^2}$ is the number of equal parts, and $\frac{a}{i^2}$ is an infinity which is relative to the infinitesimal i^2; similarly by subsequent resolutions may other infinitesimals and infinities i^3, $\frac{a}{i^3}$, ... i^n, $\frac{a}{i^n}$, be formed, which mutually involve each other.

Now although *generally* infinities and infinitesimals are symbolized respectively by ∞ and 0, yet it is manifest that all of each kind are not equal; not only do infinitesimals differ from absolute zero, but they may also differ from each other: and so may infinities differ from each other, and from a quantity which transcends every assignable quantity, that is, from absolute infinity. Hence the need of classifying such quantities.

Assuming then the order to depend on the exponent, it is plain that such *orders* must exist relatively to a certain determinate quantity, which is the subject of the exponent, and which we call the *base*. Taking therefore x to be the *base* of infinities, let x^2, x^3, ... x^n be infinities of the second, third, ... n^{th} orders; and taking i to be the base of infinitesimals,

let i^2, i^3 ...... i^n be infinitesimals of the second, third, ... n^{th} orders respectively. Similarly $\frac{a}{x}$, $\frac{a}{x^2}$, ... $\frac{a}{x^n}$ are infinitesimals of the first, second, ... n^{th} orders, if x be the infinity-base; and $\frac{a}{i}$, $\frac{a}{i^2}$, ... $\frac{a}{i^n}$ are infinities of the first, second, ... n^{th} orders, if i be the infinitesimal-base: such properties it is manifest necessarily involve each other, if i and x are so related that $x = \frac{1}{i}$, or $i = \frac{1}{x}$. These, it is to be observed, are the *definitions* of the orders of infinities and infinitesimals. Similarly, if $x^{\frac{1}{2}}$ be the infinity-base, x, $x^{\frac{3}{2}}$, x^2 ...... would be infinities of the second, third, fourth orders; and if $i^{\frac{1}{3}}$ be the infinitesimal-base, $i^{\frac{2}{3}}$, i, $i^{\frac{4}{3}}$ ...... would be infinitesimals of the second, third, fourth orders respectively.

Hence then it appears, that there will be a scale of infinities and of infinitesimals in regular sequence: such that an infinity of the n^{th} order must be infinitely subdivided to produce an infinity of the $(n-1)^{th}$ order, and infinitely quantupled to produce one of the $(n+1)^{th}$ order: infinitesimals also bear such relations to those, on either side of them in the scale, that they are infinitesimal parts of the one, and the aggregate of an infinity of the other. Thus, if x be the symbol of infinity as above, x^0 will be the symbol of the finite quantity, and the scale will be

$$x^n, \ldots\ldots x^2,\ x^1,\ x^0,\ x^{-1},\ x^{-2}, \ldots\ldots x^{-n};$$

and if i be the symbol of an infinitesimal, i^0 will represent the finite quantity, and the scale will be

$$i^{-n}, \ldots\ldots i^{-2},\ i^{-1},\ i^0,\ i^1,\ i^2, \ldots\ldots i^n,$$

the order in each scale being a descending one. Hence also, using the *general* symbols of such infinitesimals and infinities, viz. 0 and ∞, the scales become

$$0^{-n}, \ldots\ldots 0^{-2},\ 0^{-1},\ 0^0,\ 0^1,\ 0^2, \ldots\ldots 0^n,$$

$$\infty^n, \ldots\ldots \infty^2,\ \infty^1,\ \infty^0,\ \infty^{-1},\ \infty^{-2}, \ldots\ldots \infty^{-n}.$$

Thus then, although the mind is incapable of forming adequate notions of infinities and infinitesimals as they were described in

rough outline in article 5, yet they may be brought within its grasp when they are symbolized as above*. It is true that they do not always present themselves under the simple forms herein investigated, but in a subsequent chapter methods will be discussed for determining the orders of the more complex forms: and when they are so symbolized, the Calculus, as its object is, successfully considers their laws and combinations, some of which, immediately consequent upon their definitions, are stated in the following theorems, in all of which we suppose the base to be the same.

9.] THEOREM I. *Infinities and infinitesimals, like finite quantities, admit of being multiplied and divided by finite numbers, and their order is not thereby changed: but multiplication or division by the base or any power of it changes the order of the infinity and of the infinitesimal.*

Thus x^3 and $2x^3$ are infinities of the same order, and i and $\frac{i}{4}$ are infinitesimals of the same order; $i^n \times i^m = i^{n+m}$, that is, by multiplication of the base raised to a power the order of the infinitesimal is changed.

THEOR. II. *The product of an infinity and of an infinitesimal of the same order is a finite number.*

Thus $x^3 \times \frac{a}{x^3} = a$; $i \times \frac{a}{i} = a$.

THEOR. III. *The product of an infinity and of an infinitesimal of different orders is infinity or an infinitesimal, according as the order of the infinity is higher or lower than that of the infinitesimal; and the order of the product depends on the difference of the orders of the component factors.*

Thus $x^3 \times \frac{a}{x} = ax^2$; $x^n \times \frac{a}{x^m} = ax^{n-m} = \frac{a}{x^{m-n}}$, the former or latter form being taken according as n is greater or less than m[1], and therefore the result being accordingly an infinity or an infinitesimal. Similarly, $i^2 \times \frac{a}{i^4} = \frac{a}{i^2}$.

* See Poisson, Traité de Mécanique, Tome I^ere, pp. 14, 16, 2^de ed. Paris, 1833.

THEOR. IV. *The ratio to each other of two infinities or infinitesimals of the same order is finite.*

Let a and b be two finite numbers, and $\frac{a}{i^n}$, $\frac{b}{i^n}$ be two infinities of the same (viz. the n^{th}) order, then their ratio is $a : b$; similarly, if ai^n, bi^n be two infinitesimals of the n^{th} order, their ratio is $a : b$, that is, the same as before. This is also manifest geometrically. Let there be two concentric circles, the radius of one of which is double that of the other, and in them let two regular polygons of the same number of sides be described; each side of the larger is always double each side of the smaller; and as this is true whatever be the number of the sides, it is true when the number is infinitely great; in which case each side becomes infinitesimally small: and if the number in both polygons is the same, the sides are infinitesimals of the same order, and thus bear to each other the finite ratio of 2 : 1.

Hence also it follows that quantities, whose symbolical form is $\frac{0}{0}$, are indeterminate by virtue of that form, and may be either infinite, finite, or infinitesimal, and that such determinate values depend on the relation of the order of infinitesimal in the numerator to that in the denominator; that is, if the infinitesimal in the denominator be of a higher order than that in the numerator, the determinate value is infinite; if the orders are the same, the value is finite; and if that in the numerator is higher than that in the denominator, the value is infinitesimal.

Thus $\frac{(a^2-x^2)^{\frac{1}{2}}}{a-x} = \frac{0}{0}$ when $x=a$; but dividing out $(a-x)^{\frac{1}{2}}$, the result is $\frac{(a+x)^{\frac{1}{2}}}{(a-x)^{\frac{1}{2}}} = \frac{(2a)^{\frac{1}{2}}}{0} = \infty$, when $x=a$.

Similarly $\frac{a(1-x^2)}{b(1+x)} = \frac{a\times 0}{b\times 0}$, (when $x=1$), $= \frac{a(1+x)}{b} = \frac{2a}{b}$,

and $\frac{(a-x)^2}{a^2-x^2} = \frac{0}{0}$, (when $x=a$), $= \frac{a-x}{a+x} = \frac{0}{2a}$ when $x=a$.

Similar results are also manifestly true of infinities and their different orders.

THEOR. V. *The sum of two infinities or infinitesimals of the same order is the product of the infinity or infinitesimal by the sum of their coefficients; and the difference is an infinity or infinitesimal of the same order, except when the coefficients are equal, in which case it is absolutely zero.*

Thus $ai^n + bi^n = (a+b)i^n$, $ai^n - bi^n = (a-b)i^n$, $ai^n - ai^n = 0$.

THEOR. VI. *Since an infinitesimal is derived from a* FINITE *number by the resolution of the finite number into an infinity of parts, the ratio between a* FINITE *number of such infinitesimals and the original number, is that of* 0 *to* 1*; a finite number therefore of such infinitesimal parts can have no value at all when added to a finite quantity: it must be neglected.*

Thus if a and b are finite numbers, and i be an infinitesimal, of such an expression as $a + bi$, the latter part must be neglected; bi has no value at all when added to a.

THEOR. VII. *For a similar reason a finite quantity can have no value when added to an infinity, and must therefore be neglected.*

Thus of $ax + b$, the finite quantity b must be neglected, and the expression is equal to ax.

Similarly, in expressions involving the sum or difference of infinitesimals of different orders which have finite coefficients, all the higher infinitesimals must be neglected, and the lower ones alone retained. Thus let a and b be two finite quantities, and i^n and i^{n+r} two infinitesimals; then

$$ai^n + bi^{n+r} = i^n\{a + bi^r\},$$ VI

the latter part of which is equal to a by Theorem ~~IV~~; and therefore

$$ai^n + bi^{n+r} = ai^n.$$

Similarly,

$$a + bi + ci^2 + \dots\dots + ki^n = a.$$

And similarly, if an expression involves the algebraical sum of infinities of various orders, whose coefficients are finite, the expression is equal to the infinity of the highest order, and all the others, and the finite quantities, can have no value, when added to it, and must be neglected.

10.] To enable the student to appreciate the importance of the above theorems, some examples are subjoined:

Ex. 1. To find the inferior and superior limits of

$$\frac{ax^3 + bx^2 + cx + e}{mx^3 + nx^2 + px + q}.$$

By Theorem VI. the inferior limits of the numerator and denominator are severally e and q, and by Theorem VII. the superior limits are severally ax^3 and mx^3; hence the superior and inferior limits of the fraction are severally $\frac{a}{m}$ and $\frac{e}{q}$.

Ex. 2. To find the inferior and superior limits of

$$\frac{b + a^x}{a + b^x}.$$

If $x = 0$ the inferior limit becomes $\frac{b+1}{a+1}$; and if $x = \infty$, the limit is $\left(\frac{a}{b}\right)^{\infty}$, which is ∞ or 0, according as a is greater or less than b.

11.] If any one idea or conception is pregnant with the whole Calculus, it is that contained in Theorems VI and VII; they enuntiate the essential properties of infinitesimals and their reciprocal infinities: such as flow immediately from, inasmuch as they are involved in, any adequate notion of such a mode of resolution as the Calculus contemplates; were not the properties of infinitesimals such as the theorems import, the Calculus would not be what it is: from them it takes its rise, and whatever its genius be, such have they imparted to it.

On inspecting the scales of infinities and infinitesimals which are given above, it will be observed that the finite quantity is represented by the symbol which has 0 for its exponent: the reason of which by the common law of indices is manifest from the examples given in illustration of Theorem IV of Article 9; and on the correctness of thus representing it more will be said hereafter. And it will also be observed, that all the symbols on one side of it represent infinities, and all on the other infinitesimals; but it is quite arbitrary which grade shall be considered finite, or the one intermediate to the infinite and the infinitesimal. Borrowing an analogy from the senses, as

explained above, we make them the test of finiteness, but such is not necessary; and doubtless, were our senses much more delicate than they are, we should start from some order lower than we do, and call that finite which we now call infinitesimal; and if we were living amongst bodies and distances which were comparable with the distances of the fixed stars from the sun, they would doubtless be our finite quantities, and what are now finite would become infinitesimal.

12.] Thus far we have spoken of single symbols, and of their properties; it is of *continuous* variables that we shall treat, and we shall not introduce discontinuous ones without special statement. Now it is plain that two or more such variables may be combined with constants in an equation, and may be such that a change of value of one may imply a corresponding change of value of one or more of the others; when this is the case, such variables are said to depend on, and to be *functions* of, each other: and the equation which expresses the mode of dependence is said to be a *function* of such variables.

If one variable is involved in such an expression, it is said to be a function of one variable; if two variables are involved, to be a function of two variables; and so on. Thus $\sin x$, e^{ax}, $\log x$, $\sqrt{(a^2-x^2)}$, are functions of one variable, viz. x; e^{ax+by}, $\tan (ax+by)$, x^y are functions of two variables, x and y; xyz, $x^2+y^2+z^2$ are functions of three variables: similarly may we have functions of more variables. Functions are designated by the symbols F, f, ϕ, ψ, &c. Thus $\mathrm{F}(x)$ means a function of one variable x, combined or not with constants as the case may be; $\mathrm{F}(x^2)$ means a function of x^2; $\phi(x, y)$ symbolizes a function of two variables; $\psi(x, y, z)$ a function of three variables: thus these functional symbols are *general*, and the specific forms of them are the particular functions which arise from operations in algebra, trigonometry, &c. Thus if $\mathrm{F}(x) = \cos x$, F is the general symbol of an operation of which cos is the specific instance; similarly would $\tan x$, $\log x$, e^{ax}, $\sqrt{(a^2-x^2)}$, be all represented by $\mathrm{F}(x)$; and $\log (x+y)$ would be represented by $f(x, y)$.

Now as such equations represent the mode of mutual interdependence of two or more variables in their symbolized state, so in their unsymbolized state they express the relation between, and the law of, certain causes and effects.

Suppose a mass of metal to have been heated to a certain temperature, and that we have to find the temperature at any subsequent time; this latter quantity will depend on (say) these circumstances, viz. the original temperature, the law of radiation of heat, and the length of intervening time. Moreover suppose the law of relation of these four circumstances to be known, (which it is,) and it to be possible to express that law in a symbolical form; then the equation of dependence will involve four variables, viz. the original temperature, the time elapsed, the law of radiation, and the present temperature, and thus will be a function of four variables; but we shall also say, that the present temperature is a function of three other variables, and write it as follows:

Present temperature =

F (original temperature, time, law of radiation.)

Now it is possible that any one of the last three variables may vary without involving any change of the other two, in which case however the "present temperature" must vary also; and as a similar variation of any other of the three may take place, there may be three separate variations of it due to the separate variations of each of the three variables on which it depends. On this account it is called a *dependent* variable, and each of the others is called an *independent* variable.

13.] Functions are said to be *implicit* and *explicit*, according as they assume the form of one or the other of those of the last article. If by any artifice or operation, as for instance by the algebraical solution of an equation, one variable can be expressed in terms of all the others, then it is said to be an explicit function of them; but if it be not solved, and all the variables remain involved in one expression, then the function is said to be implicit. Thus the illustrating case of the last article will be an implicit function of four variables, if the quantities are combined in the form,

F (original temperature, time, law of radiation, present temperature) = 0,

and the present temperature becomes an explicit function of three variables, if written in the form,

Present temperature =

F (original temperature, time, law of radiation).

Thus $x^2+y^2-a^2=0$ is an implicit function of two variables, but $y=(a^2-x^2)^{\frac{1}{2}}$ is an explicit function of one variable, of which y is the *dependent* and x the *independent* variable; and $y=f(x)$ is the general form of such explicit functions, and $\text{F}(x,y)=c$ (c being a constant) is the general form of an implicit function of two variables. So again: $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is an implicit function of three variables of the form $\text{F}(x,y,z)=c$; whereas $z=c\left\{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right\}^{\frac{1}{2}}$, which is of the form $z=f(x,y)$, is an explicit function of two variables. Implicit functions are often written in the form,

$$u = \text{F}(x, y, z, \ldots\ldots) = c, \text{ or } = 0,$$

as the case may be.

The terms *dependent* and *independent* variables have reference to explicit functions. When functions are implicit, there are no general marks whereby to determine which of the variables may most conveniently change value first.

14.] Functions have again been divided into two classes, *algebraical* and *transcendental*: the former being those functions which involve the operations of addition, subtraction, multiplication, division, involution, and evolution, or the algebraical sum of many such functions; the latter where the operations symbolized are such as e^x, $\log_e x$, $\sin x$, $\sec^{-1} x$; that is, where they are either exponential, logarithmic, or circular. This however is a division not necessary to our present purpose.

Functions again may be *simple* or *compound*; that is, according as one or several operations (the results of which are the functions in question) are involved. Thus $y=\sin x$, $y=\log_a x$, are simple functions of x; but $y=\log\sin x$, $y=e^{\tan ax}$ are compound functions; compound functions are thus functions of functions.

It is necessary to observe, that, if two functions are represented by the same functional symbol, they are formed in the same manner by means of the variables which they involve. Thus if $f(x)=\sin x$, $f(y)=\sin y$; if $f(x)=e^{bx}$, $f(y)=e^{by}$.

15.] Functions may be either *continuous* or *discontinuous*. A continuous function is subject to the two following conditions:

1st. As the variable gradually changes, the function must gradually change.

2nd. The law symbolized by the functional character must not abruptly change.

When these two conditions are not satisfied, the function is discontinuous.

Thus, for instance, both conditions are fulfilled in the functions

$$y = ax + b, \quad y = \sin x;$$

in which, as the variable x changes, the value of the function also changes, but changes gradually, and there is no abrupt passage from one value to another; and the law symbolized by the functional character does not change, but always remains the same: but if the function were such as to express a line of the form in fig. 1, so that BA should be a continuous curve drawn after some determinate law, but at A the law suddenly should change, and the curve, from being, say, a circle, become a straight line, then the second of the above conditions is not satisfied, and the function is discontinuous. A is called a point of discontinuity. As an instance of a function of this description the following may be mentioned. Replacing the circular quantities by their exponential values, it may easily be proved that

$$\cos a + \cos (a+\beta) + \cos (a+2\beta) + \dots \text{ ad infin.} = -\frac{\sin\left(a-\frac{\beta}{2}\right)}{2\sin\frac{\beta}{2}}.$$

Suppose that $a=\frac{\beta}{2}$, then the series becomes

$$\cos a + \cos 3a + \cos 5a + \dots \text{ ad infin.} = -\frac{0}{2\sin a}:$$

but if $a =$ any multiple of π, the sum of the series assumes the indeterminate form $\frac{0}{0}$; hence we have this remarkable result, each term of the series varies continuously with a, but the sum of the series varies discontinuously, being always zero, except when a passes through some multiple of π, when the sum of the series suddenly and abruptly becomes $\frac{0}{0}$; i. e. some indeterminate quantity; thus we have a series of points of discontinuity.

It is of continuous functions of continuous variables generally that we shall treat; and if discontinuous functions are

introduced, they will be considered only for those values of the variables for which they are continuous.

16.] There are two different modes of viewing such continuous functions and variables, both of which will be convenient for the future purposes of the treatise. Firstly, suppose x_1 and x_2 to be two definite values of a variable number x, of which x_2 is the larger; and suppose the difference x_2-x_1 to be finite, and to be resolved into an infinite number of equal parts, each of which is therefore an infinitesimal; then the passage from x_1 to x_2 may be made by the successive addition of such infinitesimal elements, the whole sum of which is of course the finite quantity. Secondly, the idea of *motion* or continuous *growth* may be introduced, and we may conceive number to be in a gradually increasing state; and thus, if the *rate* of increase be finite, the increment due to a finite time will be finite, and that due to an infinitesimal interval of time will be an infinitesimal. The former mode we have hitherto invariably considered, except in the illustrations of Art. 6, but as we have now to deduce infinitesimals from finite quantities, and the latter method is more convenient for that purpose, we shall apply it. Both manifestly lead to identical quantities and results; for whereas in the one we consider a quantity during the process of generation, and the infinitesimal elements as they are successively produced; so in the other, we resolve the finite quantity when generated into its infinitesimal elements: in the latter then we arrive at the finite quantity from the elements, in the former we derive the elements from the finite quantity. The latter idea is the more complex, inasmuch as it involves motion and perhaps time, but adapts itself more readily to mechanical questions; and the former is undoubtedly best suited to geometry. It was from these two ideas that Leibnitz and Newton simultaneously, though independently, evolved the Calculus.

Thus suppose we consider the arc of a quadrant of a circle of radius a; its length is $\frac{\pi a}{2}$, which if we resolve into infinitesimal elements, each element will be the distance between two consecutive points: and the two points will be taken so near together, that the line joining them *must* be considered straight; and thus must the circle be conceived to be made up of an infinity of infinitesimal *straight* lines, and the tangent at any

point is the line which coincides with the straight line joining the point and its consecutive point; that is, the tangent is the element produced. Similarly must all continuous curves be considered as composed of infinitesimal straight lines, and all surfaces of infinitesimal plane areas, and all solids of infinitesimal elements, and all concrete bodies as made up of infinitesimal corpuscles. Or if we consider the quadrant to be generated by a point moving according to a given law, and the motion to be carried on during a finite time, and the time to be resolved into very short instants, then the space passed over in one of these instants is the infinitesimal increment of the curve; and the direction in which the point is moving at the time of generating the element is that of the tangent of the circle at the point. In this view points generate lines, lines generate surfaces, and surfaces generate solids.

17.] Consider a continuous function of one or more continuous variables. And first the simple case of an explicit function of one variable, of the form

$$y = f(x); \tag{1}$$

consider it in two successive states, and first at a finite interval apart.

Let Δ (which is used as an abbreviation of *difference*) represent a finite increment of a function or variable, so that Δx* and Δy represent the finite increments that x and y receive, and $\Delta f(x)$ the finite change in $f(x)$ due to the finite augment of the independent variable x; whence we have

$$\Delta f(x) = \Delta y = f(x+\Delta x) - f(x): \tag{2}$$

$\Delta f(x)$ is called the difference of $f(x)$, and is therefore the quantity by which $f(x)$ is increased, as the variable on which it depends is increased.

Now suppose these increments to become infinitesimal, in which case we shall use d (the abbreviation of *differential*, or small difference) to symbolize them: so that dx and dy represent the infinitesimal increments that x and y receive, and $df(x)$ the infinitesimal increment that $f(x)$ receives, owing to

* Δx may be negative; in which case it might perhaps be more properly called a decrement; but in the following treatise we shall use the words augments and increments to express the variations in the values of the variables, whether such variations cause them to increase or decrease.

the infinitesimal increment of its independent variable; thereby (2) becomes

$$d\,f(x) = dy = f(x+dx) - f(x); \tag{3}$$

$df(x)$ is called *the differential* of $f(x)$, and is therefore the infinitesimal quantity by which $f(x)$ is increased, by reason of the infinitesimal increase of the variable on which it depends.

Hence arises the name "Differential Calculus." The operation of determining the values of the differentials of the functions due to the differential increase of the variable is called Differentation, and is the first work of the Calculus; and we are said to differentiate the function *with respect to* that variable, owing to the change of which the function changes. Hereby the materials will be formed, the laws of which will be subsequently developed. We shall generally differentiate directly and without the intervention of any other symbols than those introduced above, but sometimes it is convenient to put the result in another form.

18.] Since the left-hand member of equation (3) is an infinitesimal of, say, a first order, the right-hand member must be an infinitesimal of the same order; an infinitesimal must therefore be a factor of it. But the only infinitesimal that it involves is dx, therefore we may reasonably presume that dx will be the factor; the presumption however must be verified by subsequent investigations. Dividing then both sides by dx, we have

$$\frac{dy}{dx} = \frac{d.f(x)}{dx} = \frac{f(x+dx) - f(x)}{dx}, \tag{4}$$

of which the last member is of the form $\frac{0}{0}$, but will be a finite quantity, if our presumption is correct. Let us represent it by $f'(x)$, so that

$$\frac{dy}{dx} = \frac{d.f(x)}{dx} = f'(x) \tag{5}$$

$$\therefore\ dy = d.f(x) = f'(x)\,dx: \tag{6}$$

$f'(x)$ is called *the derived function* of $f(x)$, and represents the ratio of the differential of the function to the differential of the variable; and therefore if it be known, the absolute change of the function due to the change of the variable is also known. The operation by which the derived function is determined is called "Derivation." It is under this aspect that the "Calcul des Fonctions" of *Lagrange* is constructed; $f'(x)$ is called the

differential coefficient, because it is the coefficient of dx in the equation $d.f(x) = f'(x)\,dx$.

It is also manifest, that generally

$$\frac{f(x+\Delta x)-f(x)}{\Delta x} = f'(x) + \text{R}, \tag{7}$$

when R is some residual quantity, which must be neglected when Δx becomes dx.

19.] Similarly, in considering a function of many variables, such as

$$u = \text{F}(x, y, z \,\ldots\ldots),$$

we shall use Δ and d to symbolize respectively finite and infinitesimal changes: so that Δu, Δx, $\Delta y \ldots\ldots$ represent finite, and $du, dx, dy, dz \ldots\ldots$ infinitesimal quantities; wherefore

$$\Delta u = f(x+\Delta x,\ y+\Delta y \,\ldots\ldots) - f(x, y \,\ldots\ldots) \tag{8}$$

$$du = f(x+dx,\ y+dy,\ z+dz \,\ldots\ldots) - f(x, y, z \,\ldots\ldots) \tag{9}$$

20.] Hence we may describe the Differential Calculus as follows:

The Differential Calculus is a general method and system of rules by which are determined the corresponding changes of functions and variables, when the variations of the variables are infinitesimal; and the code of laws to which they are subject, and conformably to which they may be applied to questions of Geometry and Physics.

Before however we proceed to give either general rules for the differentiation of functions, or to illustrate the process by particular examples, it is necessary to determine the values of two functions for certain values of the variables, which will be frequently applied hereafter.

21.] Lemma I. To evaluate $(1+x)^{\frac{1}{x}}$ when x is an infinitesimal.

By the Binomial Theorem

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}\,x^2 + \frac{n(n-1)\,(n-2)}{1.2.3}\,x^3 + \&c.$$

Let $n = \dfrac{1}{x}$

$$\therefore\ (1+x)^{\frac{1}{x}} = 1 + \frac{1}{x}x + \frac{1}{x}\Big(\frac{1}{x}-1\Big)\frac{x^2}{1.2} + \frac{1}{x}\Big(\frac{1}{x}-1\Big)\Big(\frac{1}{x}-2\Big)\frac{x^3}{1.2.3} + \ldots$$

$$= 1 + 1 + \frac{1-x}{1.2} + \frac{(1-x)\,(1-2x)}{1.2.3} + \ldots\ldots$$

Suppose that x were some small positive fractional number, it is plain that each factor in the numerators of the several terms of the series is less than 1; and therefore, no term being negative, the whole series is greater than its first two terms, that is, is greater than 2. Also since

$$3 = 1 + \left(1 - \frac{1}{2}\right)^{-1} = 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots\ldots$$

each term of which after the second being greater than the corresponding term in the series above, the whole series is greater; and therefore, when x is a small positive fractional number, $(1+x)^{\frac{1}{x}}$ is equal to some number greater than 2 and less than 3.

Let x be an infinitesimal, and thus be symbolized by 0, in which case by virtue of Theorem VI, Art. 9, we have

$$(1+0)^{\frac{1}{0}} = 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \frac{1}{1.2.3.4.5} + \ldots\ldots$$

which must be summed arithmetically as follows:

$$\begin{array}{rcl}
1 & = & 1\cdot \\
1 & = & 1\cdot 0000000 \\
\frac{1}{1.2} & = & \cdot 5000000 \\
\frac{1}{1.2.3} & = & \cdot 1666666 \\
\frac{1}{1.2.3.4} & = & \cdot 0416666 \\
\frac{1}{1.2\ldots4.5} & = & \cdot 0083333 \\
\frac{1}{1.2\ldots5.6} & = & \cdot 0013888 \\
\frac{1}{1.2\ldots6.7} & = & \cdot 0001984 \\
\frac{1}{1.2\ldots7.8} & = & \cdot 0000248 \\
\frac{1}{1.2\ldots8.9} & = & \cdot 0000027 \\
\hline
1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \ldots\ldots & = & 2\cdot 7182818 \ldots\ldots
\end{array}$$

and similarly, if tangents be drawn to the arc at points between A and Q and Q and P, it may be shewn that the sum of all the lines similar to SR is less than AT; but the limit of all such lines is the circular arc; therefore AT is greater than the arc.

Again, the chord AP is greater than PM, which is the sine of x, and PQ + QA is greater than AP; therefore

$$\text{PQ} + \text{QA} > \text{PM}:$$

and, drawing other chords from A and P to intermediate points on the arcs, it may be shewn that the sum of such chords is greater than the chord AP, and therefore, *à fortiori*, than PM; and as the arc AP is the limit of all such chords, it follows that it is greater than the sine PM. Therefore the arc is less than its tangent and greater than its sine.

Again, bearing in mind that $\cos x$ by its definition $= 1$, when $x = 0$, we have

$$\frac{\sin x}{\tan x} = \cos x = 1, \text{ when } x = 0;$$

whence it follows that $\sin x = \tan x$, when x is an infinitesimal; and since x is, as above shewn, always intermediate to these, it is clear that all three are equal to, and therefore may be used indifferently for, each other.

Hence, when x is an infinitesimal,

$$\sin x = x = \tan x;$$

which proposition is frequently expressed in the form "the limiting ratio of the tangent, the arc, and the sine is that of equality."

This result is also involved in the former Lemma. The exponential value of the sine gives us

$$\sin x = \frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2\sqrt{-1}},$$

$$= \frac{e^{-x\sqrt{-1}}}{2\sqrt{-1}}\left\{e^{2x\sqrt{-1}} - 1\right\}.$$

To evaluate $e^{2x\sqrt{-1}} - 1$, when x is an infinitesimal;

Let $\quad e^{2x\sqrt{-1}} - 1 = z,$

therefore x and z are simultaneously infinitesimal;

$$\therefore \; e^{2x\sqrt{-1}} = 1 + z,$$

$$\text{and } 2x\sqrt{-1} = \log_e(1+z)$$

$$= z,$$

when z is an infinitesimal, by Cor. I, Lemma I.

Hence $e^{2x\sqrt{-1}} - 1 = 2x\sqrt{-1}$;

$$\therefore \; \sin x = \frac{2x\sqrt{-1}\; e^{-x\sqrt{-1}}}{2\sqrt{-1}},$$

$$= x e^{-x\sqrt{-1}},$$

$$= x,$$

when x is an infinitesimal.

Hence also it follows that

$$\tan x = \frac{1}{\sqrt{-1}} \frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}} = x, \text{ when } x \text{ is an infinitesimal.}$$

Cor. I. When x is an infinitesimal, chord $x = x$.

For since $\text{ch}\, x = 2 \sin \frac{x}{2}$,

and by Lemma II, $\sin \frac{x}{2} = \frac{x}{2}$, when x is an infinitesimal,

$$\therefore \; \text{ch}\, x = 2\frac{x}{2} = x, \text{ when } x \text{ is an infinitesimal:}$$

that is, the chord of an infinitesimal arc is equal to the arc; which is the VIIth Lemma of the first section of Newton's Principia.

Cor. II. Hence also, if x be infinitesimal,

$$\sin^{-1} x = x = \tan^{-1} x:$$

that is, the sine and the tangent may be used indifferently for the arc, when the arc is infinitesimal.

23.] Although it may be beside our defined path, yet it is worth while to shew, that the arc and the sine are equal, when the arc is infinitesimal, only by omitting terms of a higher order, and which *must* be neglected in accordance with Theorem VI, Art. 9.

Assuming the validity of the trigonometrical proof of the series

$$\sin x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \dots\dots$$

if x is an infinitesimal, x^3, x^5, ... must be neglected as they are algebraically added to x, and we have

$$\sin x = x.$$

Also since

$$\begin{aligned} \text{versed-sine of } x &= 1 - \cos x, \\ &= 1 - \left\{1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \dots\dots\right\} \\ &= \frac{x^2}{1.2} - \frac{x^4}{1.2.3.4} + \dots\dots \end{aligned}$$

it follows that, if x be infinitesimal,

$$\text{ver-sin}\, x = \frac{x^2}{2};$$

and therefore, if the arc be an infinitesimal of the first order, the versed-sine of it is an infinitesimal of the second order. The geometrical proof of this truth is so manifest, that it is unnecessary to do more than suggest it to the student.

24.] The principles above explained are sufficient for a variety of problems, examples of which are subjoined, to give the student an insight into the *kind* of processes which he has to perform.

Ex. 1. To differentiate x^2, that is, to determine the change of x^2 due to an infinitesimal change of value of x.

$$\begin{aligned} \text{Let} \qquad y &= x^2, \\ \therefore\ y + \Delta y &= (x + \Delta x)^2, \\ \therefore\ \Delta y &= (x + \Delta x)^2 - x^2, \\ &= 2x\,\Delta x + (\Delta x)^2; \end{aligned}$$

and therefore, taking differentials instead of differences,

$$dy = d.x^2 = 2x\,dx,$$

omitting the term $(dx)^2$, which can have no value, because it is an infinitesimal of the second order, and added to one of the first order.

Hence $\dfrac{dy}{dx} = 2x,$

and therefore, if (in accordance with the notation of derived functions)

$$f(x) = x^2,$$

$$\frac{d.f(x)}{dx} = f'(x) = 2x;$$

which result, as well as others of a similar kind, as will be shewn hereafter, justifies the presumption of Art. 18.

The above process may thus be explained geometrically:

Let x represent the straight line AP, fig. 3; and suppose AP to be increased by PQ, which is represented by Δx, then from the figure it is plain that the square is increased by the rectangles DB, BQ, and the square BR, the values of which are $x \times \Delta x$, $x \times \Delta x$, $(\Delta x)^2$, whence

$$\Delta y = \Delta x^2 = 2x \, \Delta x + (\Delta x)^2.$$

Now let PQ be infinitesimal, that is, let Δx become dx, whence also Δy becomes dy, and we have

$$dy = d.x^2 = 2x \, dx + (dx)^2;$$

but $(dx)^2$ must be omitted for the following reason: $x \, dx$ symbolizes approximately a straight line, of which the length is x, and the breadth, if one may so speak, is dx; but dx^2 represents a square whose side is dx, and as dx is an infinitesimal, its square is *a point*, and as it will require an infinity of such points to make a straight line, and as the coefficient of $(dx)^2$ is not infinity, we must neglect it; that is, in calculating the enlargement of the square due to the enlargement of a side, we take account of the infinitely narrow rectangles which adjoin the sides, but must neglect the small point which is required to complete the square, and which is situated at one of the angles, as at B, and no error is committed by our so doing. Or, if we introduce the idea of motion, the enlargement of the square is due to the moving forwards of the two sides PB and CB, and the rectangles by which the square is increased are the several spaces passed over by the sides, which are the spaces contained between the lines before and after the motion; and as the spaces through which the lines have passed are very small, the lines being considered to be in two immediately *successive* positions, the small element at B becomes a point, and, as we have not an infinity of such points, the accuracy of our result is not destroyed by neglecting this small quantity; and therefore, again, the increase of the square due to the infinitesimal increase of the side is $2x \, dx$.

Ex. 2. To differentiate $\dfrac{x}{\{a^2+x^2\}^{\frac{1}{2}}}$.

Let
$$y = \frac{x}{\{a^2+x^2\}^{\frac{1}{2}}},$$

$$\therefore \quad y+\Delta y = \frac{x+\Delta x}{\{a^2+(x+\Delta x)^2\}^{\frac{1}{2}}},$$

$$\Delta y = \frac{x+\Delta x}{\{a^2+(x+\Delta x)^2\}^{\frac{1}{2}}} - \frac{x}{\{a^2+x^2\}^{\frac{1}{2}}},$$

$$= \frac{(x+\Delta x)\{a^2+x^2\}^{\frac{1}{2}} - x\{a^2+x^2+2x\Delta x+(\Delta x)^2\}^{\frac{1}{2}}}{\{a^2+x^2\}^{\frac{1}{2}}\{a^2+(x+\Delta x)^2\}^{\frac{1}{2}}};$$

and expanding the second member of the numerator by the Binomial Theorem, and neglecting the terms involving the square and higher powers of Δx, which will become infinitesimals of an order to be omitted, we have

$$\Delta y = \frac{(x+\Delta x)\{a^2+x^2\}^{\frac{1}{2}} - x\{(a^2+x^2)^{\frac{1}{2}}+x\Delta x(a^2+x^2)^{-\frac{1}{2}}\}}{\{a^2+x^2\}^{\frac{1}{2}}\{a^2+(x+\Delta x)^2\}^{\frac{1}{2}}},$$

$$= \frac{\Delta x\{a^2+x^2\}^{\frac{1}{2}} - x^2\Delta x\{a^2+x^2\}^{-\frac{1}{2}}}{\{a^2+x^2\}^{\frac{1}{2}}\{a^2+(x+\Delta x)^2\}^{\frac{1}{2}}},$$

$$= \frac{a^2\,\Delta x}{(a^2+x^2)\{a^2+(x+\Delta x)^2\}^{\frac{1}{2}}};$$

whence, taking differentials, and omitting dx, because it is added to the finite quantity x, we have

$$dy = \frac{a^2\,dx}{\{a^2+x^2\}^{\frac{3}{2}}},$$

$$\therefore \quad \frac{dy}{dx} = \frac{a^2}{\{a^2+x^2\}^{\frac{3}{2}}},$$

and $\therefore$ if $f(x) = \dfrac{x}{\{a^2+x^2\}^{\frac{1}{2}}}$, $f'(x) = \dfrac{a^2}{\{a^2+x^2\}^{\frac{3}{2}}}$.

Ex. 8. To differentiate $\dfrac{e^x}{e^x+1}$.

Let $\quad y = \dfrac{e^x}{e^x+1}$,

$$\therefore \quad y+\Delta y = \frac{e^{x+\Delta x}}{e^{x+\Delta x}+1},$$

$$\therefore \quad \Delta y = \frac{e^{x+\Delta x}}{e^{x+\Delta x}+1} - \frac{e^x}{e^x+1},$$

$$= \frac{e^{x+\Delta x}(e^x+1) - e^x(e^{x+\Delta x}+1)}{(e^{x+\Delta x}+1)(e^x+1)},$$

$$= \frac{e^x\{e^{\Delta x}-1\}}{(e^{x+\Delta x}+1)(e^x+1)}.$$

To evaluate $e^{\Delta x}-1$, when Δx becomes an infinitesimal dx.

$$\text{Let} \qquad e^{\Delta x}-1 = z,$$

$\therefore$ Δx and z are simultaneously infinitesimal.

$$\therefore \quad e^{\Delta x} = 1+z$$

$$\Delta x = \log_e (1+z)$$

$$\therefore \quad dx = z, \text{ by Cor. I. Lemma I.}$$

$$\therefore \quad e^{dx}-1 = dx;$$

replacing therefore $e^{\Delta x}-1$ by its equivalent in the above equation, and omitting Δx when added to the finite quantity x, we have

$$dy = \frac{e^x\,dx}{\{e^x+1\}^2}.$$

If therefore $$f(x) = \frac{e^x}{e^x+1},$$

$$f'(x) = \frac{e^x}{\{e^x+1\}^2}.$$

Ex. 4. To differentiate $\cos x \sin 2x$.

$$\text{Let} \qquad y = \cos x \sin 2x,$$

$$\therefore \quad y+\Delta y = \cos(x+\Delta x)\sin 2(x+\Delta x)$$

$$\begin{aligned} \therefore \quad \Delta y &= \cos(x+\Delta x)\sin 2(x+\Delta x)-\cos x \sin 2x \\ &= (\cos x \cos \Delta x-\sin x \sin \Delta x)(\sin 2x \cos 2\Delta x \\ &\qquad + \cos 2x \sin 2\Delta x)-\cos x \sin 2x; \end{aligned}$$

and taking differentials instead of differences, and therefore by reason of Lemma II replacing the sine of an infinitesimal arc by the arc itself, and the cosine by unity, we have

$$dy = (\cos x-dx \sin x)(\sin 2x+2dx \cos 2x)-\cos x \sin 2x,$$

$$dy = 2dx \cos x \cos 2x-dx \sin x \sin 2x,$$

omitting, as is necessary, the term involving $(dx)^2$.

If therefore $f(x)=\cos x \sin 2x$, $f'(x)=2 \cos x \cos 2x-\sin x \sin 2x$.

The above examples are instances of differentiating from first principles.

Ex. 5. To determine the Perimeter and the Area of a Circle of given radius.

In this and the next two examples it is especially to be borne in mind what is the strict signification of π: viz. that it is the

symbol for the incommensurable numerical quantity 3.14159..., and means *that*, and properly *that only*; and that when it is used (however incorrectly) for two right angles, it indicates *the number of times* the *essential unit angle* (that, viz. of which the subtending arc is equal to the radius) has to be taken, so as to make up two right angles; in *degrees* this unit angle is 57.29578......, a degree being the 360th part of four right angles.

Let ACD (fig. 4) be the circle, of which let the radius be a; take O the centre, and let the angle AOC be the nth part of four right angles, and AC be the side of a regular polygon of n sides inscribed in the circle; and make such a construction as is represented in the figure.

Then since $\text{AOC} = \frac{2\pi}{n}$, $\text{AOB} = \frac{\pi}{n}$,

$$\therefore \; \text{AB} = a \sin\frac{\pi}{n}, \quad \text{OB} = a\cos\frac{\pi}{n},$$

$$\therefore \; \text{perimeter of polygon} = n.\text{AC} = 2n.\text{AB} = 2na \sin\frac{\pi}{n},$$

$$\text{area of polygon} = n \times \text{area of triangle OAC} = n.\text{AB}.\text{OB}$$

$$= na^2 \sin\frac{\pi}{n} \cos\frac{\pi}{n}.$$

And when the number of the sides of the polygon is infinitely increased, the perimeter and area coincide respectively with the perimeter and area of a circle; in which case $\frac{\pi}{n}$ becomes an infinitesimal arc, and its sine must be replaced by the arc, and its cosine by unity. Hence we have

$$\text{Perimeter of circle} = 2na\frac{\pi}{n} = 2\pi a$$

$$\text{Area of circle} = na^2\frac{\pi}{n} = \pi a^2.$$

Ex. 6. To determine the convex Surface and Content of a right circular Cone.

Let A (fig. 5) be the vertex of the cone, and C be the centre of its circular base; let $\text{AC} = a$, $\text{CB} = b$, and let PQ be a side of a regular polygon of n sides circumscribing the circular base, so that $\text{PCQ} = \frac{2\pi}{n}$; whence it is plain that

$$\text{PQ} = 2\text{PB} = 2b \tan\frac{\pi}{n}.$$

Therefore the area of the triangle APQ $=$ AB $\times$ BP

$$= \{a^2+b^2\}^{\frac{1}{2}}\, b \tan\frac{\pi}{n};$$

therefore the whole convex area of the circumscribed pyramid of n sides, each of which is similar to APQ, $= nb\{a^2+b^2\}^{\frac{1}{2}} \tan\frac{\pi}{n}$.

But when n is infinitely increased, the surface of the pyramid coincides with the convex surface of the cone, and replacing $\tan\frac{\pi}{n}$ by $\frac{\pi}{n}$ in accordance with Lemma II, we have

$$\text{Convex surface of cone} = nb\{a^2+b^2\}^{\frac{1}{2}}\frac{\pi}{n}$$
$$= \pi b\{a^2+b^2\}^{\frac{1}{2}};$$

that is, the convex surface of a Cone is equal to the product of the slant side by the semi-circumference of the base.

Hence, whole surface of Cone $= \pi b^2 + \pi b\{a^2+b^2\}^{\frac{1}{2}}$.

Again, by Euclid XII. 7, the content of the pyramid APQC is one-third of that of the prism of the same altitude and the same base; and therefore

$$\text{Content of circumscribed pyramid} = \frac{n}{3}.\ \text{AC. CB. BP}$$
$$= \frac{n}{3}\, ab^2 \tan\frac{\pi}{n}.$$

Let the number of the sides of the polygon circumscribing the base be infinitely increased, in which case the pyramid assumes the limiting form of the cone, then $\frac{\pi}{n}$ becomes an infinitesimal arc, so that it must replace $\tan\frac{\pi}{n}$, and we have

$$\text{Content of Cone} = \frac{n}{3}\, ab^2\, \frac{\pi}{n}$$
$$= \frac{\pi ab^2}{3}.$$

Now it is plain that πab^2 is the content of a cylinder of altitude a described on the circular base whose area is πb^2;

hence it follows, that the content of a cone is one-third of that of its circumscribing cylinder.

Ex. 7. To determine the Surface and Content of a Sphere of given radius.

Suppose the circle ACBD (fig. 6) to be a plane section of the sphere by the paper, or to be that by the revolution of which, about its diameter BOA, the sphere is generated. Let the radius $= a$, and be divided into n equal parts, each of which is therefore equal to $\frac{a}{n}$, and suppose MN to be one of these parts,

$$\therefore \quad \text{MN} = \frac{a}{n}.$$

Draw the ordinates MP, NQ, including between them the arc PQ of the circle. Then, if PQ be infinitesimal, and which therefore must be considered straight, the *zone* of the sphere generated by the revolution of PQ about AO is equal to a rectangle, of which one side is PQ and the other is the circular path described by P, as it revolves; that is,

$$\text{The surface of the zone} = \text{PQ} \times 2\pi \times \text{MP}.$$

$$\text{Let POA} = \theta; \quad \therefore \quad \text{PQ} = \text{MN} \operatorname{cosec}\theta = \frac{a}{n\sin\theta}$$

$$\text{MP} = a\sin\theta$$

$$\therefore \quad \text{Surface of zone} = \frac{2\pi a^2}{n}.$$

And since the surface of the whole sphere is equal to $2n$ times the surface of such a zone, we have

$$\text{Surface of sphere} = 4\pi a^2;$$

and is equal therefore to four times the area of a great circle of the sphere.

Now suppose a cylinder to be described about the sphere, and to be of the same altitude as the sphere, and to be generated by the revolution of the rectangle AEFB about the diameter BA; and suppose two planes drawn through M and N, perpendicular to BOA, so as to intercept a zone of the cylinder whose height is P′Q′ or MN: then, since a is the radius of the cylinder, the surface of the intercepted zone $= 2\pi a \times \text{MN}$; but from above,

$$\text{Corresponding surface of zone of sphere} = 2\pi a\frac{a}{n} = 2\pi a \times \text{MN},$$

Hence, if a cylinder be described about a given sphere, and the surface of the cylinder be divided into zones by planes perpendicular to its axis, the surfaces of the intercepted zones of the cylinder and the sphere are equal; and therefore,

The whole surface of the sphere is equal to the convex surface of its circumscribing cylinder.

Again, to determine the Content of a Sphere.

Let a regular polyhedron of n equal faces be described about the sphere;

Let $S =$ area of each face, and $a =$ radius of sphere,

then the content of each pyramid, whose vertex is at the centre of the sphere and base is S, is $\frac{aS}{3}$: and therefore the content of the circumscribed polyhedron is $n\frac{aS}{3}$; but when the number of faces is infinitely increased so that the polyhedron ultimately coincides with the sphere, nS becomes the surface of the sphere, and is therefore equal to $4\pi a^2$; therefore the content of the sphere $= \frac{4\pi a^3}{3}$.

Ex. 8. To determine the Area of a Parabola.

Let OPQBA (fig. 7) be an area bounded by the Parabola OPB, the axis OA, and the ordinate AB which is parallel to the tangent at the vertex O.

$$\text{Let} \qquad \left.\begin{aligned} \text{OM} &= x \\ \text{MP} &= y \end{aligned}\right\} \quad \left.\begin{aligned} \text{OA} &= a \\ \text{AB} &= b \end{aligned}\right\}$$

$$\therefore \text{ the equation to the curve is } y^2 = \frac{b^2}{a} x.$$

Let AB be divided into n equal parts, and through the several points of division let lines be drawn parallel to OA; let K and L be respectively the rth and $(r+1)$th points of division, and therefore $\text{KL} = \frac{b}{n}$; then, if n be taken very large, the area of $\text{PQLK} = \text{KL} \times \text{KP}$.

$$\begin{aligned} \text{But} \qquad \text{KP} &= \text{OA} - \text{OM} \\ &= a - \frac{a}{b^2}\text{MP}^2 \end{aligned}$$

$$\text{KP} = a - \frac{a}{b^2}\left(r\frac{b}{n}\right)^2$$

$$= a - \frac{r^2}{n^2}a$$

$$\therefore \text{ area PQKL} = \frac{b}{n}\left\{a - \frac{r^2}{n^2}a\right\}$$

$$= \frac{ab}{n}\left\{1 - \frac{r^2}{n^2}\right\};$$

and as this expression is true of every slice similar to PQKL, the whole area is the sum of all such, which are obtained by giving to r severally the values 1, 2, 3, ... $(n-1)$, n; therefore

$$\text{Area of parabola} = \frac{ab}{n}\left\{1 - \frac{1^2}{n^2} + 1 - \frac{2^2}{n^2} + 1 - \frac{3^2}{n^2} + \ldots + 1 - \frac{n^2}{n^2}\right\}$$

$$= \frac{ab}{n}\left\{n - \frac{1^2 + 2^2 + \ldots\ldots + n^2}{n^2}\right\}$$

$$= \frac{ab}{n}\left\{n - \frac{(2n-1)\,n(n+1)}{6n^2}\right\}$$

$$= ab\left\{1 - \frac{2n^3 + n^2 - n}{6n^3}\right\}$$

$$= \frac{2}{3}\,ab;$$

when n becomes infinity.

Hence the Area of such a Parabola is equal to two-thirds of the circumscribing rectangle.

CHAPTER II.

CONSTRUCTION OF RULES FOR THE DIFFERENTIATION OF FUNCTIONS.

25.] If a constant be connected with a function of a variable by the symbol of addition or subtraction, it disappears in the process of differentiation.

Let
$$\begin{aligned} y &= f(x) \pm c, \\ y+\Delta y &= f(x+\Delta x) \pm c: \\ \therefore \quad \Delta y &= f(x+\Delta x) - f(x), \\ dy &= f(x+dx) - f(x), \\ &= d.f(x), \\ &= f'(x)\, dx, \qquad \text{by Art. 18.} \end{aligned}$$

This result is manifest from the very nature of constants, which do not admit of increase, and therefore have the same value in both states in which the function is considered, and therefore disappear in the subtraction; thus we may say, that the differential of a constant is zero.

Ex. 1.
$$\begin{aligned} y &= a+x, \\ dy &= dx. \end{aligned}$$

Ex. 2.
$$\begin{aligned} y &= x-a, \\ dy &= dx. \end{aligned}$$

26.] A constant connected with a function of a variable by the processes of multiplication or division is not changed by differentiation.

Let
$$\begin{aligned} y &= cf(x), \\ y+\Delta y &= cf(x+\Delta x), \\ \Delta y &= c\{f(x+\Delta x) - f(x)\}, \\ dy &= c\{f(x+dx) - f(x)\}, \\ &= cd.f(x), \\ &= cf'(x)\, dx. \end{aligned}$$

Or again:

Let $$y = \frac{1}{c} f(x),$$

$$\Delta y = \frac{1}{c}\{f(x+\Delta x) - f(x)\},$$

$$dy = \frac{1}{c}\{f(x+dx) - f(x)\},$$

$$= \frac{1}{c}\, d.f(x),$$

$$= \frac{1}{c} f'(x)\, dx.$$

Suppose $c = -1$, then $y = -f(x)$,

$$dy = -d.f(x),$$

$$= -f'(x)\, dx.$$

Ex. 1. $$y = ax + b,$$
$$dy = a dx.$$

Ex. 2. $$y = \frac{x}{a} - c,$$
$$dy = \frac{dx}{a}.$$

27.] The differentiation of an algebraical sum of functions of variables.

Let $$y = f(x) \pm F(x) \pm \phi(x) + \dots\dots$$

$$y + \Delta y = f(x+\Delta x) \pm F(x+\Delta x) \pm \phi(x+\Delta x) + \dots\dots$$

$$\Delta y = f(x+\Delta x) - f(x) \pm \{F(x+\Delta x) - F(x)\}$$
$$\pm \{\phi(x+\Delta x) - \phi(x)\} + \dots\dots$$

$$\therefore\ dy = f(x+dx) - f(x) \pm \{F(x+dx) - F(x)\}$$
$$\pm \{\phi(x+dx) - \phi(x)\} \pm \dots\dots$$

$$= d.f(x) \pm d.F(x) \pm d.\phi(x) \pm \dots\dots$$

$$= f'(x)dx \pm F'(x)\, dx \pm \phi'(x)\, dx \pm \dots\dots$$

Hence the differential of the algebraical sum of functions is equal to the sum of the differentials of the functions.

Ex. 1. $$y = ax - bf(x) + c,$$
$$\therefore\ dy = adx - bf'(x)\, dx.$$

From the last two Articles it is plain, that if the function to be differentiated be of the form

$$y = f(x) + \sqrt{(-1)}\,\phi(x),$$

one of the functions being what is commonly called impossible,

$$\begin{aligned} dy &= d.f(x) + \sqrt{(-1)}\,d.\phi(x), \\ &= f'(x)\,dx + \sqrt{(-1)}\,\phi'(x)\,dx; \end{aligned}$$

whence it appears that, in the differentiation of impossible quantities, we may treat the symbol of impossibility in the same way as we treat an ordinary constant or symbol of affection.

28.] Differentiation of a Product of two Functions of a Variable.

Let $\quad y = f(x) \times \phi(x),$

$$\therefore\ y + \Delta y = f(x+\Delta x) \times \phi(x+\Delta x),$$

$$\therefore\ \Delta y = f(x+\Delta x) \times \phi(x+\Delta x) - f(x) \times \phi(x);$$

whence, adding and subtracting the required quantities,

$$\begin{aligned} y = {} & \{f(x+\Delta x) - f(x)\}\,\phi(x) + \{\phi(x+\Delta x) - \phi(x)\}\,f(x) \\ & + \{f(x+\Delta x) - f(x)\}\,\{\phi(x+\Delta x) - \phi(x)\} \end{aligned}$$

$$\therefore\ dy = d.f(x) \times \phi(x) + d.\phi(x) \times f(x) + d.f(x) \times d.\phi(x);$$

and omitting the last term, which is an infinitesimal of the second order, we have

$$\begin{aligned} dy &= d.f(x) \times \phi(x) + d.\phi(x) \times f(x), \\ &= f'(x)\,\phi(x)\,dx + \phi'(x) f(x)\,dx. \end{aligned}$$

Hence it follows that, the differential of the product of two functions is the sum of the products of each function and the differential of the other.

Ex. 1. $\quad y = (b + cx)(e - ax),$

$$\begin{aligned} \therefore\ dy &= c\,dx(e - ax) - a\,dx(b + cx), \\ &= (ec - ab - 2acx)\,dx. \end{aligned}$$

29.] Differentiation of the Quotient of two Functions.

Let $\quad y = \dfrac{f(x)}{\phi(x)},$

$$y + \Delta y = \frac{f(x+\Delta x)}{\phi(x+\Delta x)};$$

$$\therefore\ \Delta y = \frac{f(x+\Delta x)}{\phi(x+\Delta x)} - \frac{f(x)}{\phi(x)},$$

$$= \frac{f(x+\Delta x)\,\phi(x) - f(x)\,\phi(x+\Delta x)}{\phi(x)\,\phi(x+\Delta x)},$$

$$= \frac{\{f(x+\Delta x) - f(x)\}\,\phi(x) - \{\phi(x+\Delta x) - \phi(x)\}f(x)}{\phi(x)\,\phi(x+\Delta x)};$$

$$\therefore \quad dy = \frac{d.f(x) \times \phi(x) - d.\phi(x) \times f(x)}{\{\phi(x)\}^2},$$

omitting the infinitesimal dx when added to the finite quantity x in the denominator; and therefore

$$dy = \frac{f'(x)\,\phi(x)\,dx - \phi'(x)\,f(x)\,dx}{\{\phi(x)\}^2}.$$

Ex. 1. $\quad y = \dfrac{a+bx}{b+ax},$

$$dy = \frac{bdx\,(b+ax) - adx\,(a+bx)}{(b+ax)^2},$$

$$= \frac{b^2-a^2}{(b+ax)^2}\,dx.$$

30.] Differentiation of x^n.

Let

$$y = x^n,$$

$$y + \Delta y = (x+\Delta x)^n,$$

$$\therefore \quad \Delta y = (x+\Delta x)^n - x^n,$$

$$= x^n \left\{ \left(1+\frac{\Delta x}{x}\right)^n - 1 \right\}.$$

Let

$$\left(1+\frac{\Delta x}{x}\right)^n - 1 = z,$$

$\therefore$ Δx and z are simultaneously infinitesimal;

$$\therefore \quad \left(1+\frac{\Delta x}{x}\right)^n = 1+z,$$

$$n \log_e \left(1+\frac{\Delta x}{x}\right) = \log_e (1+z).$$

Let Δx and z become infinitesimal, then by Cor. I, Lemma I,

$$n\frac{dx}{x} = z;$$

therefore when Δx is infinitesimal, $\left(1+\dfrac{\Delta x}{x}\right)^n - 1$ is to be replaced by $n\dfrac{dx}{x}$:

$$\therefore \quad dy = x^n n \frac{dx}{x},$$

$$= nx^{n-1}\,dx,$$

that is $\qquad d.x^n = nx^{n-1}\,dx.$

Therefore to differentiate x^n we have the following rule:

Multiply by the exponent, diminish the exponent by unity, and multiply by dx.

Suppose the exponent to be negative, then

$$y = x^{-n},$$

$$dy = -nx^{-(n+1)}\,dx;$$

and suppose $n = -1$,

$$y = \frac{1}{x} = x^{-1},$$

$$dy = -x^{-2}\,dx = -\frac{dx}{x^2}.$$

Suppose the index to be fractional, $n = \frac{p}{q}$,

$$y = x^{\frac{p}{q}},$$

$$dy = \frac{p}{q}\,x^{\frac{p}{q}-1}\,dx.$$

Also, since $\frac{dy}{dx} = nx^{n-1}$, it is manifest that if $f(x) = x^n$ $f'(x) = nx^{n-1}$; that is, the derived function of x^n is nx^{n-1}.

The above rule for differentiating x^n might also have been determined as follows:

$$y = x^n,$$

$$y + \Delta y = (x+\Delta x)^n;$$

$$\therefore \quad \Delta y = x^n + \frac{n}{1}x^{n-1}\Delta x + \frac{n(n-1)}{1.2}x^{n-2}(\Delta x)^2 + \ldots\ldots - x^n$$

$$= \frac{n}{1}x^{n-1}\Delta x + \frac{n(n-1)}{1.2}x^{n-2}(\Delta x)^2 + \ldots\ldots$$

Let the increments be infinitesimal, and we have

$$dy = nx^{n-1}\,dx.$$

Ex. 1. $\qquad y = a + bx^3 \qquad \therefore \quad dy = 3bx^2\,dx.$

Ex. 2. $\qquad y = a\sqrt{x} \qquad dy = \frac{adx}{2\sqrt{x}}.$

Ex. 3. $y = c+\frac{b}{\sqrt{x}}$ $\qquad dy = \frac{b}{2x^{\frac{3}{2}}}$.

Ex. 4. $y = \frac{a}{x^2}$ $\qquad dy = \frac{2adx}{x^3}$.

Ex. 5. $y = (a+bx^n)(c-ex^m)$

$$dy = nbx^{n-1}\,dx\,(c-ex^m)-mex^{m-1}\,dx\,(a+bx^n).$$

Ex. 6. $y = \frac{x^2}{a^2+x^2}$,

$$dy = \frac{2x\,dx\,(a^2+x^2)-2x\,dx\,x^2}{(a^2+x^2)^2},$$

$$= \frac{2a^2\,x\,dx}{(a^2+x^2)^2}.$$

31.] Differentiation of a Compound Function, or of a Function of a Function.

Thus far we have calculated the change of value of a function of x, due to a small variation of x; but suppose the function to be compound, and thus to depend on x, not immediately, but by means of some function; what modification must the principles and results of the preceding Articles undergo? The only condition to which x is subject is, that it is a *continuous* and finite variable for the values assigned to it; and this condition will be fulfilled, if x be replaced by any *continuous* and finite function of x, as its variation due to a small variation of x will be infinitesimal: in this case therefore the original function will vary in consequence of the variation of this latter function; and *pari ratione* this latter function may be a compound function, and therefore not vary immediately by reason of the variation of its variable, but only through some other function; and so on to any number of functions. Taking then our former notation, in this case y is a function of a function of x, or a function of many functions of x, and its differential must be calculated as follows:

Since if $y = f(x)$, $dy = f'(x)\,dx$.

Therefore, if x be replaced by $\phi(x)$, dx must be replaced by $\phi'(x)\,dx$; and therefore if

$$y = f\{\phi(x)\},$$

$$dy = f'\{\phi(x)\}\,\phi'(x)\,dx.$$

And so again, if the x involved in $\phi(x)$ were a function of x, say $\psi(x)$, $\phi(x)$ must be replaced by $\phi\{\psi(x)\}$, and $\phi'(x)\,dx$ by $\phi'\{\psi(x)\}\,\psi'(x)\,dx$; and a similar notation will adapt itself to a function of a function ...... (to n functions) of x. Thus if

$$y = f\{\phi[\psi(x)]\},$$
$$dy = f'\{\phi[\psi(x)]\}\,\phi'[\psi(x)]\,\psi'(x)\,dx.$$

Or we may substitute as follows:

$$\begin{aligned} \text{Let}\quad \psi(x) &= z & \therefore\ dz &= \psi'(x)\,dx,\\ \phi[\psi(x)] &= \phi(z) = u & \therefore\ du &= \phi'(z)\,dz,\\ \therefore\ f\{\phi[\psi(x)]\} &= f(u) = y & \therefore\ dy &= f'(u)\,du;\end{aligned}$$

$$\begin{aligned}\therefore\ dy &= f'(u)\,du,\\ &= f'(u)\,\phi'(z)\,dz,\\ &= f'(u)\,\phi'(z)\,\psi'(x)\,dx,\\ &= \frac{dy}{du}\,\frac{du}{dz}\,\frac{dz}{dx}\,dx.\end{aligned}$$

But as generally it is more convenient to make as few substitutions as possible, the former value of dy is preferable to the latter; the latter also, it is to be observed, is an identity.

Hence, by reason of the last Article, if

$$y = \{f(x)\}^n,$$
$$dy = n\{f(x)\}^{n-1} f'(x)\,dx.$$

Ex. 1. $y = f(x \pm a)$ $\qquad dy = f'(x \pm a)\,dx.$

Ex. 2. $y = f(cx)$ $\qquad dy = cf'(cx)\,dx.$

Ex. 3. $y = f(ax^n \pm b)$ $\qquad dy = anf'(ax^n \pm b)\,x^{n-1}\,dx.$

Ex. 4. $y = (ax^n + c)^m$ $\qquad dy = m(ax^n + c)^{m-1}\,nax^{n-1}\,dx.$

Ex. 5. $y = (a + bx^3)^2\,(a - x)^{\frac{3}{2}}$,

$$dy = (a + bx^3)\,(a - x)^{\frac{1}{2}}\,\frac{12ab^2 - 18bx^3 - 3a}{2}\,dx.$$

Ex. 6. $y = \dfrac{(x+3)^3}{(x^2-2)^2}$,

$$dy = \frac{3(x+3)^2\,(x^2-2)\,dx - 4x\,(x^2-2)\,(x+3)^3\,dx}{(x^2-2)^4},$$
$$= \frac{(x+3)^2\,(-x^2+12x-6)}{(x^2-2)^3}\,dx.$$

32.] Differentiation of a^x.

Let
$$y = a^x,$$
$$y+\Delta y = a^{x+\Delta x},$$
$$\therefore \quad \Delta y = a^{x+\Delta x}-a^x,$$
$$= a^x\{a^{\Delta x}-1\}.$$

Let
$$a^{\Delta x}-1 = z,$$

$\therefore$ Δx and z are simultaneous infinitesimals;
$$\therefore \quad a^{\Delta x} = z+1,$$
$$\Delta x \log_e a = \log_e(1+z),$$

and by Cor. I, Lemma I, $\log_e(1+z)=z$, when z is infinitesimal,
$$\therefore \quad dx \log_e a. = z;$$

and therefore $a^{dx}-1$ must be replaced by $dx \log_e a$,
$$\therefore \quad dy = a^x \log_e a.\ dx,$$
$$d.a^x = \log_e a.\ a^x\ dx:$$

that is, the differential of a^x is the product of the quantity of the Napierian logarithm of a, and of the differential of the exponent.

And hence, if $a^x = f'(x)$, $f'(x) = \log_e a.\ a^x$.

33.] Differentiation of e^x.

In the last Article for the base a write the Napierian base e; in which case $\log_e a$ becomes $\log_e e$, which is equal to 1, and
$$\therefore \quad d.e^x = e^x\ dx:$$

that is, the differential of e^x is the product of the quantity and the differential of the exponent.

Hence, if $f(x) = e^x$, $f'(x) = e^x$, and e^x is a quantity which is equal to its own derived function.

And such results are true by reason of Article 31, when x is replaced by any function of x; hence we have
$$d.\ a^{f(x)} = \log_e a.\ a^{f(x)} f'(x)\ dx,$$
$$d.\ e^{f(x)} = e^{f(x)} f'(x)\ dx.$$

Ex. 1. $y = a^{cx}$ $\qquad dy = \log_e a.\ a^{cx}\ cdx.$

Ex. 2. $y = e^{(b+cx^2)}$ $\qquad dy = e^{(b+cx^2)}\ 2cx\ dx.$

Ex. 3. $y = e^{-x^n}$ $\qquad dy = -e^{-x^n}\ nx^{n-1}\ dx.$

Ex. 4. $y = e^{x\sqrt{-1}}+e^{-x\sqrt{-1}}$ $\qquad dy = \sqrt{-1}\{e^{x\sqrt{-1}}-e^{-x\sqrt{-1}}\}\,dx.$

Ex. 5. $y = e^x(x^2-2x+2)$,

$$dy = e^x\{x^2-2x+2\}\,dx + e^x\{2x-2\}\,dx,$$
$$= e^x x^2\,dx.$$

Ex. 6. $y = e^{e^x}$ $\qquad dy = e^{e^x}.\,e^x.\,dx.$

34.] Differentiation of $\log_a x$.

$$\begin{aligned} \text{Let} \qquad y &= \log_a x, \\ y+\Delta y &= \log_a(x+\Delta x), \\ \Delta y &= \log_a(x+\Delta x) - \log_a x, \\ &= \log_a\left(\frac{x+\Delta x}{x}\right), \\ &= \log_a\left(1+\frac{\Delta x}{x}\right); \end{aligned}$$

but when $\frac{\Delta x}{x}$ becomes an infinitesimal, viz. $\frac{dx}{x}$, $\log_a\left(1+\frac{\Delta x}{x}\right)$ $= \frac{dx}{x\log_e a}$ by Cor. II, Lemma I; hence

$$dy = \frac{dx}{x\log_e a},$$

$$\therefore \quad d.\log_a x = \frac{dx}{x\log_e a};$$

and therefore if $\qquad f(x) = \log_a x,$

$$f'(x) = \frac{1}{x\log_e a}.$$

Hence also $\qquad d.\log_a f(x) = \dfrac{f'(x)\,dx}{\log_e a\,f(x)}.$

35.] Differentiation of $\log_e x$.

Suppose the base a of the last Article to be e, then $\because$ $\log_e e = 1$.

$$d.\log_e x = \frac{dx}{x};$$

and therefore if $\qquad f(x) = \log_e x,$

$$f'(x) = \frac{1}{x}.$$

Hence also $\qquad d.\log_e f(x) = \dfrac{f'(x)\,dx}{f(x)}.$

Hence the differential of the Napierian logarithm of any function of x is equal to the differential of the function divided by the function itself.

36.] The general result of Art. 34 might also have been found from that of Art. 32, as follows:

$$y = \log_a x,$$
$$\therefore \quad a^y = x,$$
$$\therefore \quad \log_e a \, . \, a^y \, dy = dx,$$
$$\therefore \quad dy = \frac{dx}{x \log_e a}.$$

Ex. 1. $\quad y = \log x^n \qquad dy = \dfrac{nx^{n-1}\,dx}{x^n} = n\dfrac{dx}{x}.$

Ex. 2. $\quad y = \log (x + \sqrt{1+x^2}) \qquad dy = \dfrac{dx}{\{1+x^2\}^{\frac{1}{2}}}.$

Ex. 3. $\quad y = \log \log x \qquad dy = \dfrac{d.\log x}{\log x} = \dfrac{dx}{x \log x}.$

Ex. 4. $\quad y = x^2 \log x \qquad dy = 2x\,dx \log_e x + x dx,$

$$= \{2 \log_e x + 1\} \, x dx.$$

Ex. 5.
$$y = f(x) \times \phi(x),$$
$$\therefore \quad \log_e y = \log_e f(x) + \log_e \phi(x),$$
$$\frac{dy}{y} = \frac{f'(x)\,dx}{f(x)} + \frac{\phi'(x)\,dx}{\phi(x)},$$
$$\therefore \quad dy = f'(x) \times \phi(x)\,dx + \phi'(x) f(x)\,dx.$$

Ex. 6.
$$y = \{f(x)\}^{\phi(x)},$$
$$\log_e y = \phi(x) \log_e f(x),$$
$$\frac{dy}{y} = \phi'(x)\,dx. \log_e f(x) + \phi(x)\frac{f'(x)\,dx}{f(x)},$$
$$\therefore \quad dy = \{f(x)\}^{\phi(x)} \left\{ \phi'(x) \log_e f(x) + \phi(x)\frac{f'(x)}{f(x)} \right\} dx.$$

Ex. 7.
$$y = x^n,$$
$$\log_e y = n \log_e x,$$
$$\frac{dy}{y} = n\frac{dx}{x},$$
$$\therefore \quad dy = nx^{n-1}\,dx;$$

whence we have an independent proof of the result of Art. 30.

37.] The Differentiation of Functions of x connected with one another by Multiplication or Division.

$$y = f(x) \times \text{F}(x) \times \phi(x) \times \Phi(x) \times \ldots\ldots$$

If all the functions are positive, we have

$$\log y = \log f(x) + \log \text{F}(x) + \log \phi(x) + \&\text{c}.$$

$$\therefore \frac{dy}{y} = \frac{d.f(x)}{f(x)} + \frac{d.\text{F}(x)}{\text{F}(x)} + \frac{d.\phi(x)}{\phi(x)} + \&\text{c}.$$

$$dy = d\{f(x)\,\text{F}(x)\,\phi(x)\ldots\},$$

$$= f(x)\text{F}(x)\phi(x)\ldots\left\{\frac{d.f(x)}{f(x)} + \frac{d.\text{F}(x)}{\text{F}(x)} + \frac{d.\phi(x)}{\phi(x)} + \cdots\right\}.$$

Or, if some of the functions be negative,

$$y^2 = [f(x)]^2\,[\text{F}(x)]^2\,[\phi(x)]^2\ \&\text{c}.$$

$$\log(y^2) = \log[f(x)]^2 + \log[\text{F}(x)]^2 + \log[\phi(x)]^2 + \&\text{c}.$$

$$\frac{dy}{y} = \frac{d.f(x)}{f(x)} + \frac{d.\text{F}(x)}{\text{F}(x)} + \frac{d.\phi(x)}{\phi(x)} + \&\text{c}.$$

$$\therefore dy = f(x)\text{F}(x)\phi(x)\ldots\left\{\frac{d.f(x)}{f(x)} + \frac{d.\text{F}(x)}{\text{F}(x)} + \frac{d.\phi(x)}{\phi(x)} + \cdots\right\}.$$

Therefore, in either case,

$$d\{f(x)\text{F}(x)\phi(x)\ldots\} = d.f(x)\text{F}(x)\phi(x)\ldots + d.\text{F}(x)f(x)\phi(x)\ldots$$
$$+ d.\phi(x)f(x)\,\text{F}(x)\ldots + \&\text{c}.$$

And, if $$y = \frac{f(x)}{\phi(x)},$$

$$\log(y^2) = \log[f(x)]^2 - [\log\phi(x)]^2;$$

$$\therefore \frac{dy}{y} = \frac{d.f(x)}{f(x)} - \frac{d.\phi(x)}{\phi(x)},$$

$$\therefore dy = d\left\{\frac{f(x)}{\phi(x)}\right\} = \frac{f(x)}{\phi(x)}\left\{\frac{d.f(x)}{f(x)} - \frac{d.\phi(x)}{\phi(x)}\right\},$$

$$= \frac{d.f(x)\phi(x) - d.\phi(x)f(x)}{\{\phi(x)\}^2},$$

$$= \frac{f'(x)\,\phi(x)\,dx - \phi'(x)f(x)\,dx}{\{\phi(x)\}^2};$$

which is the same result as that obtained in Art. 29.

In the latter cases we have raised both sides to the square

power, in order to avoid the apparent difficulty of taking the logarithms of negative quantities; but had this not been done, we should have had $\log_e(-1)$, which will be shewn in the next Chapter to be equal to $(2k+1)\pi\sqrt{-1}$, (see Art. 62) the differential of which is zero, as it is a constant quantity.

Ex. 1.
$$y = (a+x)(b+2x)(c+3x),$$
$$dy = dx\,(b+2x)(c+3x)+2dx\,(c+3x)(a+x) + 3dx\,(a+x)(b+2x).$$

Ex. 2.
$$y = e^x\left\{\frac{x+1}{x-1}\right\}^{\frac{1}{2}},$$
$$\therefore\ \log_e y = x+\frac{1}{2}\log(x+1)-\frac{1}{2}\log(x-1),$$
$$\frac{dy}{y} = dx+\frac{1}{2}\frac{dx}{x+1}-\frac{1}{2}\frac{dx}{x-1};$$
$$\therefore\ dy = y\left\{dx-\frac{dx}{x^2-1}\right\},$$
$$= y\left\{\frac{x^2-2}{x^2-1}\right\}dx,$$
$$= e^x\frac{(x^2-2)\,dx}{(x+1)^{\frac{1}{2}}(x-1)^{\frac{3}{2}}}.$$

And a complicated product or quotient may often be most easily differentiated by first taking the logarithms, and then differentiating and reducing.

38.] Differentiation of (α) $\sin x$, (β) $\cos x$, (γ) $\tan x$, (δ) $\sec x$, (ϵ) $\operatorname{versin} x$, (ζ) $\cot x$, (η) $\operatorname{cosec} x$.

(α)
$$y = \sin x,$$
$$y+\Delta y = \sin(x+\Delta x);$$
$$\therefore\ \Delta y = \sin(x+\Delta x)-\sin x,$$

and since
$$\sin A-\sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2},$$
$$\therefore\ \Delta y = 2\cos\left(x+\frac{\Delta x}{2}\right)\sin\frac{\Delta x}{2}.$$

If then the increments are infinitesimal, replacing $\sin\frac{dx}{2}$ by

$\frac{dx}{2}$ in accordance with Lemma II, and omitting $\frac{dx}{2}$ which is added to x, we have

$$dy = 2 \cos x \frac{dx}{2};$$

$$\therefore \quad d.\sin x = \cos x.\, dx.$$

If therefore
$$f(x) = \sin x,$$
$$f'(x) = \cos x.$$

Hence also
$$d.\sin f(x) = \cos f(x).f'(x)\, dx.$$

(β)
$$y = \cos x,$$
$$y + \Delta y = \cos(x + \Delta x);$$
$$\therefore \quad \Delta y = \cos(x + \Delta x) - \cos x,$$
$$= 2 \sin\left(x + \frac{\Delta x}{2}\right) \sin\frac{\Delta x}{2},$$

since
$$\cos A - \cos B = -2 \sin\frac{A+B}{2} \sin\frac{A-B}{2},$$

$$\therefore \quad dy = -2 \sin x \frac{dx}{2},$$

$$\therefore \quad d.\cos x = -\sin x\, dx.$$

If therefore
$$f(x) = \cos x,$$
$$f'(x) = -\sin x.$$

Hence also
$$d.\cos f(x) = -\sin f(x).f'(x)\, dx.$$

(γ)
$$y = \tan x,$$
$$y + \Delta y = \tan(x + \Delta x);$$
$$\therefore \quad \Delta y = \tan(x + \Delta x) - \tan x,$$
$$= \frac{\tan x + \tan \Delta x}{1 - \tan x \tan \Delta x} - \tan x,$$
$$= \frac{\tan \Delta x \{1 + (\tan x)^2\}}{1 - \tan x \tan \Delta x}.$$

Let the increments be infinitesimal, in which case $\tan dx$ must be replaced by dx by reason of Lemma II; therefore

$$dy = \frac{\{1 + (\tan x)^2\}\, dx}{1 - \tan x.\, dx},$$
$$= \{1 + (\tan x)^2\}\, dx,$$

omitting the term of the denominator involving dx, inasmuch as it is an infinitesimal subtracted from a finite quantity;

$$\therefore \quad d.\tan x = \{1+(\tan x)^2\}\, dx = (\sec x)^2\, dx.$$

Hence therefore if $f(x) = \tan x, \quad f'(x) = (\sec x)^2.$

Also $\quad d.\tan f(x) = \{\sec f(x)\}^2 . f'(x)\, dx.$

(δ) $$y = \sec x = \frac{1}{\cos x} = (\cos x)^{-1};$$

$$\therefore \quad dy = (\cos x)^{-2} \sin x\, dx, \text{ by Art. 30,}$$

$$\therefore \quad dy = d.\sec x = \frac{\sin x\, dx}{(\cos x)^2} = \sec x \tan x\, dx.$$

If therefore $\quad f(x) = \sec x,$

$$f'(x) = \sec x . \tan x;$$

and $\quad d.\sec f(x) = \sec f(x). \tan f(x). f'(x)\, dx.$

(ε) $$y = \text{versin}\, x = 1 - \cos x;$$

$$\therefore \quad dy = d.(1 - \cos x) = \sin x\, dx.$$

If therefore $\quad f(x) = \text{versin}\, x,$

$$f'(x) = \sin x;$$

and $\quad d.\text{versin} f(x) = \sin f(x). f'(x)\, dx.$

(ζ) $$y = \cot x,$$

$$y + \Delta y = \cot(x + \Delta v),$$

$$\Delta y = \cot(x+\Delta x) - \cot x,$$

$$= \frac{1 - \tan x \tan \Delta x}{\tan x + \tan \Delta x} - \frac{1}{\tan x},$$

$$= \frac{-\{1 + (\tan x)^2\} \tan \Delta x}{(\tan x + \tan \Delta x) \tan x},$$

$$\therefore \quad dy = -\frac{(\sec x)^2\, dx}{(\tan x)^2},$$

$$= -\frac{dx}{(\sin x)^2} = -(\text{cosec}\, x)^2\, dx.$$

If therefore $\quad f(x) = \cot x,$

$$f'(x) = -(\text{cosec}\, x)^2;$$

and $\quad d.\cot f(x) = -\{\text{cosec} f(x)\}^2 f'(x)\, dx.$

$$(\eta) \qquad y = \operatorname{cosec} x,$$
$$= \frac{1}{\sin x} = (\sin x)^{-1},$$
$$dy = -(\sin x)^{-2} \cos x \, dx,$$
$$= -\frac{\cos x}{(\sin x)^2} dx,$$
$$d.\operatorname{cosec} x = -\operatorname{cosec} x \cot x \, dx.$$

If therefore
$$f(x) = \operatorname{cosec} x,$$
$$f'(x) = -\operatorname{cosec} x. \cot x;$$
and
$$d.\operatorname{cosec} f(x) = -\operatorname{cosec} f(x) \cot f(x). f'(x) \, dx.$$

39.] It is also manifest from the geometry of the figure (see fig. 8), that the increments of the trigonometrical functions due to the increment of the arc are such as have been deduced in the above formulæ. For let AP be the arc of a circle whose radius is unity, and PQ be any small arc added to it:

Let $\quad$ AP $= x$, PQ $= \Delta x$.

Then PM $= \sin x$, NQ $= \sin(x+\Delta x)$; $\therefore$ QR $= \Delta.\sin x$.
CM $= \cos x$, CN $= \cos(x+\Delta x)$; $\therefore$ NM $= -\Delta.\cos x$.
AT $= \tan x$, AT′ $= \tan(x+\Delta x)$; $\therefore$ TT′ $= \Delta.\tan x$.
CT $= \sec x$, CT′ $= \sec(x+\Delta x)$; $\therefore$ ST′ $= \Delta.\sec x$.

ST : QP :: CT : CP, i.e. ST : Δx :: $\sec x$: 1.

$\therefore$ ST $= \Delta x \sec x$.

When Δx becomes dx, PQ, ST become straight lines, and perpendicular to CT, and ST becomes $dx \sec x$: whence

$d.\sin x =$ QR $=$ PQ $\sin$ QPR $=$ PQ $\cos$ RPC $=$ PQ $\cos$ PCA
$= dx \cos x$.

$d.\cos x = -$NM $= -$PR $= -$PQ $\cos$ QPR $= -$PQ $\sin$ PCA
$= -dx \sin x$.

$d.\tan x =$ TT′ $=$ ST $\sec$ T′TS $=$ ST cosec CTA $=$ ST $\sec$ PCA
$= (\sec x)^2 \, dx$.

$d.\sec x =$ ST′ $=$ ST $\tan$ T′TS $=$ ST $\tan$ PCA $= \sec x \tan x \, dx$.

40.] The differentials of $\tan x$ and $\cot x$ may also be determined from those of $\sin x$ and $\cos x$ as follows:

$$y = \tan x = \frac{\sin x}{\cos x};$$

therefore, by Art. 29, $$dy = \frac{\cos x\, d.\sin x - \sin x\, d.\cos x}{(\cos x)^2},$$

$$dy = \frac{(\sin x)^2\, dx + (\cos x)^2\, dx}{(\cos x)^2},$$

$$= \frac{dx}{(\cos x)^2},$$

$$= (\sec x)^2\, dx.$$

And $$y = \cot x = \frac{\cos x}{\sin x},$$

$$\therefore\ dy = d.\cot x = \frac{\sin x\, d.\cos x - \cos x\, d.\sin x}{(\sin x)^2},$$

$$= \frac{-(\sin x)^2\, dx - (\cos x)^2\, dx}{(\sin x)^2},$$

$$= \frac{-dx}{(\sin x)^2},$$

$$= -(\text{cosec}\, x)^2\, dx.$$

41.] Ex. 1. $y = \sin x^2$ $\quad dy = \cos x^2\, 2x\, dx.$

Ex. 2. $y = \sin\sin x$ $\quad dy = \cos\sin x \cos x\, dx.$

Ex. 3. $y = a^{\cos x}$ $\quad dy = -\log_e a\, a^{\cos x} \sin x\, dx.$

Ex. 4. $$y = (\sin x + \cos x)^n,$$
$$dy = n(\sin x + \cos x)^{n-1} (\cos x - \sin x)\, dx.$$

Ex. 5. $$y = e^{ax} \cos nx,$$
$$dy = ae^{ax} \cos nx\, dx - ne^{ax} \sin nx\, dx,$$
$$= e^{ax} \{a \cos nx - n \sin nx\}\, dx.$$

Ex. 6. $$y = \frac{(\sin x)^n}{\sin nx},$$

$$dy = \frac{n(\sin x)^{n-1} \cos x \sin nx\, dx - n(\sin x)^n \cos nx\, dx}{(\sin nx)^2},$$

$$= \frac{n(\sin x)^{n-1} (\cos x \sin nx - \sin x \cos nx)\, dx}{(\sin nx)^2},$$

$$= \frac{n(\sin x)^{n-1} \sin (n-1)\, x\, dx}{(\sin nx)^2}.$$

42.] **Differentiation of Inverse circular Functions.**

(a) $$y = \sin^{-1}\frac{x}{a},$$

$$y+\Delta y = \sin^{-1}\frac{x+\Delta x}{a},$$

$$\therefore\ \Delta y = \sin^{-1}\frac{x+\Delta x}{a} - \sin^{-1}\frac{x}{a},$$

$$= \sin^{-1}\left\{\frac{x+\Delta x}{a}\left(1-\frac{x^2}{a^2}\right)^{\frac{1}{2}} - \frac{x}{a}\left(1-\left\{\frac{x+\Delta x}{a}\right\}^2\right)^{\frac{1}{2}}\right\},$$

$$= \sin^{-1}\left\{\frac{x+\Delta x}{a}\left(1-\frac{x^2}{a^2}\right)^{\frac{1}{2}} - \frac{x}{a}\left(1-\frac{x^2}{a^2} - \frac{2x\,\Delta x}{a^2}\right)^{\frac{1}{2}}\right\},$$

$$= \sin^{-1}\left\{\frac{x+\Delta x}{a}\left(1-\frac{x^2}{a^2}\right)^{\frac{1}{2}} - \frac{x}{a}\left(1-\frac{x^2}{a^2}\right)^{\frac{1}{2}} + \frac{x^2\Delta x}{a^3}\left(1-\frac{x^2}{a^2}\right)^{-\frac{1}{2}}\right\},$$

$$\therefore\ dy = \sin^{-1}\left\{\frac{dx}{a}\left(1-\frac{x^2}{a^2}\right)^{\frac{1}{2}} + \frac{x^2\,dx}{a^3}\left(1-\frac{x^2}{a^2}\right)^{-\frac{1}{2}}\right\},$$

$$= \frac{dx}{a}\left(1-\frac{x^2}{a^2}\right)^{\frac{1}{2}} + \frac{x^2\,dx}{a^3}\left(1-\frac{x^2}{a^2}\right)^{-\frac{1}{2}},$$

by reason of Cor. II, Lemma II,

$$\therefore\ dy = d.\sin^{-1}\frac{x}{a} = \frac{dx}{\{a^2-x^2\}^{\frac{1}{2}}}.$$

If therefore $$f(x) = \sin^{-1}\frac{x}{a},$$

$$f'(x) = \frac{1}{\{a^2-x^2\}^{\frac{1}{2}}}.$$

The differential may also be found as follows:

$$y = \sin^{-1}\frac{x}{a},$$

$$\therefore\ x = a\sin y,$$

$$dx = a\cos y\,dy;$$

but $$\cos y = \{1-(\sin y)^2\}^{\frac{1}{2}} = \left\{1-\frac{x^2}{a^2}\right\}^{\frac{1}{2}}:$$

$$\therefore \quad dx = a\left\{1-\frac{x^2}{a^2}\right\}^{\frac{1}{2}} dy = \{a^2-x^2\}^{\frac{1}{2}}\, dy,$$

$$\therefore \quad dy = d.\sin^{-1}\frac{x}{a} = \frac{dx}{\{a^2-x^2\}^{\frac{1}{2}}};$$

and as this latter process is the shorter, we shall use it in the following similar problems, although the former would be more elementary and is equally applicable.

(β)
$$y = \cos^{-1}\frac{x}{a},$$

$$\therefore \quad \cos y = \frac{x}{a},$$

$$\therefore \quad d.\cos y = -\sin y\, dy = \frac{dx}{a};$$

but
$$\sin y = \{1-(\cos y)^2\}^{\frac{1}{2}} = \left\{1-\frac{x^2}{a^2}\right\}^{\frac{1}{2}} = \frac{1}{a}\{a^2-x^2\}^{\frac{1}{2}},$$

$$\therefore \quad -\{a^2-x^2\}^{\frac{1}{2}}\, dy = dx,$$

$$\therefore \quad dy = d.\cos^{-1}\frac{x}{a} = \frac{-dx}{\{a^2-x^2\}^{\frac{1}{2}}}.$$

If therefore
$$f(x) = \cos^{-1}\frac{x}{a},$$

$$f'(x) = \frac{-1}{\{a^2-x^2\}^{\frac{1}{2}}}.$$

(γ)
$$y = \tan^{-1}\frac{x}{a},$$

$$\therefore \quad x = a\tan y,$$

$$dx = a\{1+(\tan y)^2\}\, dy,$$

$$= a\left(1+\frac{x^2}{a^2}\right) dy,$$

$$\therefore \quad dy = d.\tan^{-1}\frac{x}{a} = \frac{a\, dx}{a^2+x^2}.$$

If therefore
$$f(x) = \tan^{-1}\frac{x}{a},$$

$$f'(x) = \frac{a}{a^2+x^2}.$$

(δ)
$$y = \sec^{-1}\frac{x}{a},$$
$$\therefore \quad x = a\sec y,$$
$$dx = a\sec y\tan y\, dy,$$
$$= a\frac{x}{a}\left\{\frac{x^2}{a^2}-1\right\}^{\frac{1}{2}} dy,$$
$$= \frac{x}{a}\{x^2-a^2\}^{\frac{1}{2}}\, dy:$$
$$\therefore \quad dy = d.\sec^{-1}\frac{x}{a} = \frac{a\, dx}{x\{x^2-a^2\}^{\frac{1}{2}}}.$$

If therefore
$$f(x) = \sec^{-1}\frac{x}{a},$$
$$f'(x) = \frac{a}{x\{x^2-a^2\}^{\frac{1}{2}}}.$$

(ϵ)
$$y = \operatorname{versin}^{-1}\frac{x}{a},$$
$$x = a\operatorname{versin} y,$$
$$dx = a\sin y\, dy;$$

but
$$\sin y = \{1-\cos^2 y\}^{\frac{1}{2}} = \{1-(1-\operatorname{versin} y)^2\}^{\frac{1}{2}},$$
$$= \left\{1-\left(1-\frac{x}{a}\right)^2\right\}^{\frac{1}{2}},$$
$$= \frac{1}{a}\{2ax-x^2\}^{\frac{1}{2}},$$
$$\therefore \quad dx = \{2ax-x^2\}^{\frac{1}{2}}\, dy,$$
$$\therefore \quad dy = d.\operatorname{versin}^{-1}\frac{x}{a} = \frac{dx}{\{2ax-x^2\}^{\frac{1}{2}}}.$$

If therefore
$$f(x) = \operatorname{versin}^{-1}\frac{x}{a},$$
$$f'(x) = \frac{1}{\{2ax-x^2\}^{\frac{1}{2}}}.$$

(ζ) Similarly, if $y = \cot^{-1}\dfrac{x}{a}$,
$$dy = d.\cot^{-1}\frac{x}{a} = \frac{-a\, dx}{a^2+x^2}.$$

(η)
$$y = \operatorname{cosec}^{-1}\frac{x}{a},$$
$$dy = d.\operatorname{cosec}^{-1}\frac{x}{a} = \frac{-a\,dx}{x\{x^2-a^2\}^{\frac{1}{2}}}.$$

It is to be observed in respect of the above differentials, that the sum of $d.\sin^{-1}\frac{x}{a}$ and $d.\cos^{-1}\frac{x}{a}$ is zero, which arises from the fact that the sum of two such arcs is $\frac{\pi}{2}$, which is constant, and whose differential therefore is equal to zero. The same thing, and for the same reason, is true of $d.\tan^{-1}\frac{x}{a} + d.\cot^{-1}\frac{x}{a}$, and of $d.\sec^{-1}\frac{x}{a} + d.\operatorname{cosec}^{-1}\frac{x}{a}$.

43.] Hence, if the radius be unity, we have

$$d.\sin^{-1}x = \frac{dx}{\{1-x^2\}^{\frac{1}{2}}},$$
$$d.\cos^{-1}x = \frac{-dx}{\{1-x^2\}^{\frac{1}{2}}},$$
$$d.\tan^{-1}x = \frac{dx}{1+x^2},$$
$$d.\cot^{-1}x = \frac{-dx}{1+x^2},$$
$$d.\sec^{-1}x = \frac{dx}{x\{x^2-1\}^{\frac{1}{2}}},$$
$$d.\operatorname{cosec}^{-1}x = \frac{-dx}{x\{x^2-1\}^{\frac{1}{2}}},$$
$$d.\operatorname{versin}^{-1}x = \frac{dx}{\{2x-x^2\}^{\frac{1}{2}}}.$$

Hence also, by reason of Art. 31,

$$d.\sin^{-1}f(x) = \frac{f'(x)\,dx}{\{1-(f(x))^2\}^{\frac{1}{2}}},$$
$$d.\tan^{-1}f(x) = \frac{f'(x)\,dx}{1+(f(x))^2}.$$

Similar results are true of the other functions.

44.] These results may also be obtained from geometry, by a process similar to that employed in Art. 39. See fig. 8.

To find $d.\sin^{-1}x$.

$$\text{Let } \quad \text{MP} = x \qquad \therefore \ \text{AP} = \sin^{-1}x = y,$$
$$\text{NQ} = x+\Delta x \qquad \therefore \ \text{AQ} = \sin^{-1}(x+\Delta x) = y+\Delta y.$$
$$\therefore \ \text{RQ} = \Delta x.$$
$$\Delta y = \Delta\sin^{-1}x = \text{the arc PQ} = \text{RQ} \sec \text{PQR} = \text{RQ} \sec \text{PCA};$$
$$\therefore \ dy = d.\sin^{-1}x = dx\sec(\sin^{-1}x) = \frac{dx}{(1-x^2)^{\frac{1}{2}}}.$$

To find $d.\tan^{-1}x$.

$$\text{AT} = x \qquad \therefore \ \text{AP} = \tan^{-1}x = y,$$
$$\text{AT}' = x+\Delta x \qquad \text{AQ} = \tan^{-1}(x+\Delta x) = y+\Delta y,$$
$$\therefore \ \text{TT}' = \Delta x.$$
$$\therefore \ \Delta y = \Delta\tan^{-1}x = \text{the arc PQ} = \text{ST}\frac{\text{CP}}{\text{CT}}:$$
$$\because \ \text{PQ} : \text{ST} :: \text{CP} : \text{CT};$$

but
$$\text{ST} = \text{TT}'\cos\text{STT}' = \Delta x\cos\tan^{-1}x = \frac{\Delta x}{\{1+x^2\}^{\frac{1}{2}}},$$
$$\text{CP} = \text{CA} = 1,$$
$$\text{CT} = \sec\tan^{-1}x = \{1+x^2\}^{\frac{1}{2}}:$$
$$\therefore \ d.\tan^{-1}x = \frac{dx}{1+x^2};$$

and similarly may the other differentials be found.

45.] Ex. 1. $\quad y = \sin^{-1}mx \qquad dy = \dfrac{mdx}{\{1-m^2x^2\}^{\frac{1}{2}}}.$

Ex. 2. $\quad y = e^{\tan^{-1}x} \qquad dy = e^{\tan^{-1}x}\dfrac{dx}{1+x^2}.$

Ex. 3. $\quad y = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right) \qquad dy = \dfrac{2dx}{1+x^2}.$

Ex. 4. $\quad y = x^{\sin^{-1}x},$

$$\therefore \ \log_e y = \sin^{-1}x\log x.$$
$$\frac{dy}{y} = \log x\frac{dx}{\{1-x^2\}^{\frac{1}{2}}} + \frac{\sin^{-1}x.\,dx}{x},$$
$$\therefore \ dy = x^{\sin^{-1}x}\left\{\frac{x\log x+(1-x^2)^{\frac{1}{2}}\sin^{-1}x}{x(1-x^2)^{\frac{1}{2}}}\right\}dx.$$

The number of examples which have been differentiated, including the types of all known algebraical, exponential, logarithmic, and circular forms, and all of which indicate the derived function to be a new function of x, is sufficient to justify the presumption of Art. 18. It is in fact almost an inductive argument *per simplicem enumerationem*.

46.] On Functions of many Variables.

When many variables are involved in an equation, it is theoretically possible so to arrange them that one should be a function of all the others, whereby it becomes an explicit function of many variables; and these latter variables may be independent of each other, or have such relations amongst them that a variation of one may involve a variation of one or more of the others, whereby the number of independent variables may be diminished; the former case, which is the more general, shall be first considered, and then such modifications shall be introduced into the result as shall adapt it to the latter, and as the mutual interdependence of the variables may require.

For the sake of illustration let us consider a tree, and suppose its growth to depend on three circumstances which are independent of each other, viz. the fertility of the soil, the rain that waters it, and the heat of the sun: then, if the relation or law of connexion of these four things be expressed mathematically and in an explicit form, we shall have the growth of the tree a function of three variables; thus,

Growth of tree = F (fertile soil, rain, solar heat).

And if the law be known which connects the single effect with the three producing causes, then F (the symbol of the form of the function) is known.

Now observing the independence of the three variables involved under the functional symbol, and that therefore any one may vary without necessitating a variation of the others, and that a variation of any one will cause a variation in the tree's growth, it follows that there may be three several variations of the effect, owing to the separate and several variations of the three producing causes. Thus, suppose the fertility of the soil to be increased, while the other two causes are unchanged, the tree's growth may be thereby increased; and similarly may it be increased by a change in either of the other two causes. Now although these three acting causes are so

mbined by means of the connecting equation, that when a ite change of one has taken place, a subsequent change of other may not have the same *absolute* effect on the tree's owth as if it had changed without the first having previously ried, yet such will not be the case when the changes are finitesimal; because the infinitesimal variation of one acting bsequently to and on the back of the infinitesimal variation the other, infinitesimals of a higher order, or products of finitesimals, will be introduced which must be neglected by rtue of Theorem VI, Art. 9; and thus the absolute infinisimal change in the tree's growth, due to the infinitesimal riation of any one of the three acting causes, will be the me, whether the other two causes have changed or not. And erefore, when all three have changed, the resulting change of e effect will be the sum of the three effects due to each of e three producing causes; of this however a mathematical d rigorous proof will be given subsequently. Thus we have riations of the growth of four distinct kinds: a variation due each of the three causes acting separately, and the whole riation which is the sum of these three. This latter is called *total variation* or a *total differential*, and the former are called *rtial variations* or *partial differentials*; each of which, it is ain, is to be calculated on the supposition, that one variable ly changes and that the others do not change.

Or again: consider a plane rectangle OPRS (see fig. 9), of hich one side OP $= x$, and the other OS $= y$; and let u represent its area. Therefore,

$$u = xy.$$

ow the area may vary owing to a change in the length of ther side, or to changes of the lengths of both. Suppose x be increased by PQ $= \Delta x$, (see Art. 19); then the area is creased by the rectangle RPQW $= y\,\Delta x$; which, when Δx comes dx, is $y\,dx$. Or suppose y to receive a finite increent, that is, OS to be increased by ST $= \Delta y$: then the increase the area is the rectangle TR $= x\,\Delta y$; which, when the increase is infinitesimal, is $x\,dy$; hence the partial changes or crements of the rectangle are $y\,dx$ and $x\,dy$. But suppose th sides to vary simultaneously, that is, x to become $x + dx$, d y to become $y + dy$, then the rectangle becomes

$$(x + dx)\,(y + dy) = xy + y\,dx + x\,dy + dx\,dy;$$

that is, the increment of the rectangle is $y\,dx + x\,dy + dx\,dy$, which severally represent the rectangles RQ, TR, UW; and therefore, when the increments are infinitesimal, the first two are rectangles of finite length but of infinitesimal breadth, and the latter, being a rectangle with infinitesimal sides, is but a *point*, which must therefore be neglected, and we have the total differential of u

$$= xdy + ydx$$
$$= \text{the sum of the partial differentials.}$$

Bearing in mind what was said in Art. 19, and that Greek letters are used to signify finite changes or differences, and English letters infinitesimal variations or differentials, we shall find the following symbolization convenient.

Let $u = \text{F}(x, y, z, \ldots\ldots)$ be the function of many variables, and $\text{D}u$ or DF represent the total differential of u due to the variations of all the variables,

$d_x u$, or $d_x \text{F}$ . . the partial change of u due to a change of x,

$d_y u$, or $d_y \text{F}$ . . . . y,

$d_z u$, or $d_z \text{F}$ . . . . z,

and let $\left(\frac{du}{dx}\right)$, $\left(\frac{du}{dy}\right)$, $\left(\frac{du}{dz}\right)$, ... (omitting the subscript letters) represent the ratios to the increments of the variables, of the several variations of the function due to the variation of the variables separately; that is, let them represent *partial* differential coefficients, or derived functions, the brackets indicating that they do so, and the variable in the denominator of the fractions being that, due to the change of which, the partial variation of u is calculated.

Thus, although the numerators involve the same symbol du, yet the same thing is not represented by them; for the du arising from the growth of x into $x + dx$, may be a wholly different function from the du which comes from the growth of y into $y + dy$. If then we have occasion to use these symbols together, a mark of distinction is necessary, such as $d_x u$, $d_y u$, ... but when the ratios of the changes are required, the denominator in addition to the bracket is sufficient to indicate the origin of the function.

47.] Differentiation of a Function of Two Variables.

$$u = \text{F}(x, y).$$

Let x and y receive the increments Δx and Δy, and the corresponding increment of u be Δu; so that

$$\Delta u = \text{F}(x+\Delta x, y+\Delta y) - \text{F}(x, y);$$

$$\therefore\ \Delta u = \text{F}(x+\Delta x, y+\Delta y) - \text{F}(x, y+\Delta y) + \text{F}(x, y+\Delta y) - \text{F}(x, y).$$

Let the variations of x and y be very small, then

Δu becomes $\text{D}u$, which is the total change in u;

$\text{F}(x+\Delta x, y+\Delta y) - \text{F}(x, y+\Delta y)$ becomes $d_x u$,

because whatever difference there is between the first and second of the above quantities, it is solely due to the change of x; and for a similar reason

$\text{F}(x, y+\Delta y) - \text{F}(x, y)$ becomes $d_y u$, and therefore

$$\text{D}u = d_x u + d_y u;$$

that is, the total differentiation of a function of two variables is equal to the sum of the partial differentials. Also, we have

$$\Delta u = \frac{\text{F}(x+\Delta x, y+\Delta y) - \text{F}(x, y+\Delta y)}{\Delta x}\,\Delta x + \frac{\text{F}(x, y+\Delta y) - \text{F}(x, y)}{\Delta y}\,\Delta y;$$

$$\therefore\ \text{D}u = \left(\frac{du}{dx}\right) dx + \left(\frac{du}{dy}\right) dy.$$

In which formula dx and dy are the small increments of x and y; whence we easily find the ratio of the total change of the function to the change of either variable: thus we have

$$\frac{\text{D}u}{dx} = \left(\frac{du}{dx}\right) + \left(\frac{du}{dy}\right)\frac{dy}{dx},$$

$$\frac{\text{D}u}{dy} = \left(\frac{du}{dx}\right)\frac{dx}{dy} + \left(\frac{du}{dy}\right).$$

Ex. 1.
$$\begin{aligned} u &= ay^2 + bxy + cx^2 + ey + gx + k, \\ d_x u &= by\,dx + 2cx\,dx + g\,dx, \\ &= (by + 2cx + g)\,dx, \\ d_y u &= 2ay\,dy + bx\,dy + e\,dy, \\ &= (2ay + bx + e)\,dy; \end{aligned}$$

$$\therefore \quad \mathrm{D}u = d_x u + d_y u,$$

$$= (by + 2cx + g)\,dx + (bx + 2ay + e)\,dy.$$

Or $\quad \left(\frac{du}{dx}\right) = by + 2cx + g,$

$$\left(\frac{du}{dy}\right) = bx + 2ay + e;$$

$$\therefore \quad \mathrm{D}u = \left(\frac{du}{dx}\right) dx + \left(\frac{du}{dy}\right) dy,$$

$$= (by + 2cx + g)\,dx + (bx + 2ay + e)\,dy.$$

Ex. 2. $\quad u = \frac{x^2}{a^2} + \frac{y^2}{b^2},$

$$\left(\frac{du}{dx}\right) = \frac{2x}{a^2}, \quad \left(\frac{du}{dy}\right) = \frac{2y}{b^2};$$

$$\therefore \quad \mathrm{D}u = \frac{2x}{a^2}\,dx + \frac{2y}{b^2}\,dy.$$

Ex. 3. $\quad u = \sin^{-1}\frac{x}{a} + \sin^{-1}\frac{y}{b},$

$$\left(\frac{du}{dx}\right) = \frac{1}{\{a^2-x^2\}^{\frac{1}{2}}} \qquad \left(\frac{du}{dy}\right) = \frac{1}{\{b^2-y^2\}^{\frac{1}{2}}};$$

$$\therefore \quad \mathrm{D}u = \frac{dx}{\{a^2-x^2\}^{\frac{1}{2}}} + \frac{dy}{\{b^2-y^2\}^{\frac{1}{2}}}.$$

Ex. 4. $\quad u = xy\,\phi\left(\frac{y}{x}\right),$

$$\left(\frac{du}{dx}\right) = y\,\phi\left(\frac{y}{x}\right) - \frac{y^2}{x}\,\phi'\left(\frac{y}{x}\right), \text{ by virtue of Art. 81.}$$

$$\left(\frac{du}{dy}\right) = x\,\phi\left(\frac{y}{x}\right) + y\,\phi'\left(\frac{y}{x}\right);$$

$$\therefore \quad \mathrm{D}u = (y\,dx + x\,dy)\,\phi\left(\frac{y}{x}\right) + \frac{y(x\,dy - y\,dx)}{x}\,\phi'\left(\frac{y}{x}\right)$$

Ex. 5. $\quad u = x\,\phi(xy),$

$$\left(\frac{du}{dx}\right) = \phi(xy) + xy\,\phi'(xy),$$

$$\left(\frac{du}{dy}\right) = x^2\,\phi'(xy);$$

$$\therefore \quad \mathrm{D}u = \phi(xy)\,dx + x\,(y\,dx + x\,dy)\,\phi'(xy).$$

48.] Differentiation of an Implicit Function of Two Variables.

The principles applied in the last and preceding Articles enable us to differentiate an implicit function of two variables, and thereby to determine the ratio of the corresponding differentials of the variables, without the expression being put in the form of an explicit function of one variable*.

* To the above method of differentiating implicit functions it may be objected, that if

$$u = f(x, y) = c,$$

there can be no change in u due to a partial change in x, because u is constant; and for a similar reason there can be no variation in u due to a variation of y only; and therefore that x and y cannot vary separately, but must vary simultaneously; that one cannot change without the other, and therefore that there can be no partial variations of the function. To this it is replied by asking, What do we mean by a constant such as c on the right-hand side of the equation? We mean that whose total variation is zero. If therefore we consider c to be the sum of two quantities c_1 and c_2, such that when x varies c_1 varies, and c_2 varies when y varies, but that c_1 and c_2 are so related that $dc_1 + dc_2 = 0 = dc$; then the total variation of $f(x, y)$ being the sum of the partial variations will be equal to 0, and the condition of $f(x, y)$ being equal to c will be satisfied. Whereas then each partial variation taken separately is a violation of the condition expressed by the equation, yet the relation of the two when added together is such, as to be in accordance with the equation. The partial variation of the second neutralises the inconsistency of the partial variation of the first with the equation. This may be geometrically explained as follows. Suppose

$$f(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} = c,$$

which equation represents an ellipse the size of which depends on the value of the constant c, as well as on a and b. Suppose we consider y alone to vary, while x is constant, the point corresponding to $x, y + dy$, is no longer on the original ellipse, but lies a little above it or below it, according to the sign of dy; i.e. it is a point of another ellipse, which second ellipse depends on the variation of the part of c corresponding to the variation of y. And now suppose x to vary, y being constant, in consequence of which the other part of c, which has a variation due to the variation of x, changes, but changes in such a manner as to bring back the point to the original ellipse, the variations of the two parts of the constant being such that the sum of them is zero; whereas then the partial variation of the first carried the point off the ellipse, that of the second brought it back again. This case then is exactly that of the total variation of a function of two variables, except that the two partial variations are so related as to render the total variation equal to zero.

Let $f(x, y) = c$ be the implicit function of two variables, which may be put in the form

$$u = f(x, y) = c.$$

Then, since u is equal to a constant, $\mathrm{D}u = 0$; and therefore by the last Article,

$$\left(\frac{du}{dx}\right) dx + \left(\frac{du}{dy}\right) dy = 0;$$

$$\therefore \quad \frac{dy}{dx} = -\frac{\left(\frac{du}{dx}\right)}{\left(\frac{du}{dy}\right)}.$$

Ex. 1. $\quad u = x^3 + 3axy + y^3 = a^3,$

$$\left(\frac{du}{dx}\right) = 3x^2 + 3ay \qquad \left(\frac{du}{dy}\right) = 3y^2 + 3ax;$$

$$\therefore \quad \frac{dy}{dx} = -\frac{x^2 + ay}{y^2 + ax}.$$

Ex. 2. $\quad u = x^y + y^x = a,$

$$\left(\frac{du}{dx}\right) = yx^{y-1} + y^x \log_e y \qquad \left(\frac{du}{dy}\right) = x^y \log_e x + xy^{x-1}.$$

$$\therefore \quad \frac{dy}{dx} = -\frac{yx^{y-1} + y^x \log y}{xy^{x-1} + x^y \log x}.$$

49.] Differentiation of a Function of many Variables, all of which are independent of each other.

Let $\qquad u = \mathrm{F}(x, y, z, \ldots\ldots)$

be a function of many variables, $x, y, z, \ldots\ldots$ all of which are independent of each other; then

$$\begin{aligned}
u + \Delta u &= \mathrm{F}(x+\Delta x,\ y+\Delta y,\ z+\Delta z, \ldots);\\
\therefore\ \Delta u &= \mathrm{F}(x+\Delta x,\ y+\Delta y,\ z+\Delta z, \ldots) - \mathrm{F}(x, y, z, \ldots)\\
&= \mathrm{F}(x+\Delta x,\ y+\Delta y,\ z+\Delta z, \ldots) - \mathrm{F}(x, y+\Delta y, z+\Delta z, \ldots)\\
&\quad + \mathrm{F}(x,\ y+\Delta y,\ z+\Delta z, \ldots) \quad - \mathrm{F}(x,\ y,\ z+\Delta z, \ldots)\\
&\quad + \mathrm{F}(x,\ y,\ z+\Delta z, \ldots) \qquad - \ldots\ldots\\
&\quad + \ldots\ldots\ldots\ldots \qquad - \ldots\ldots\\
&\quad + \ldots\ldots\ldots\ldots \qquad - \mathrm{F}(x, y, z \ldots);
\end{aligned}$$

whence, making the increments to be infinitesimal, and observing the forms in which the last lines have been written, we have

$$\text{D}u = \text{D}\text{F} = d_x u + d_y u + d_z u + \ldots\ldots$$

that is, the total differential is equal to the sum of the partial differentials.

And if the result be expressed in terms of partial derived functions, as in Art. 46, then

$$\text{D}u = \text{D}\text{F} = \left(\frac{du}{dx}\right) dx + \left(\frac{du}{dy}\right) dy + \left(\frac{du}{dz}\right) dz + \ldots\ldots$$

Whence, as in Art. 47, the ratio of the total differential to that of any one of the variables may be found.

Ex. 1. $u = \sin(yz + zx + xy)$,

$$\left(\frac{du}{dx}\right) = (z + y)\cos(yz + zx + xy),$$

$$\left(\frac{du}{dy}\right) = (x + z)\cos(yz + zx + xy),$$

$$\left(\frac{du}{dz}\right) = (y + x)\cos(yz + zx + xy);$$

$$\therefore\ \text{D}u = \cos(yz+zx+xy)\,\{(y+z)\,dx + (z+x)\,dy + (x+y)\,dz\}.$$

Ex. 2. $u = (x^2+y^2+z^2)^{\frac{1}{2}} + \tan^{-1}\dfrac{x}{z} + ax+by+cz;$

$$\therefore\ \text{D}u = \frac{xdx+ydy+zdz}{(x^2+y^2+z^2)^{\frac{1}{2}}} + \frac{zdx-xdz}{x^2+z^2} + adx + bdy + cdz.$$

Ex. 3. $u = \dfrac{x^2 y}{z^2-a^2}$,

$$\left(\frac{du}{dx}\right) = \frac{2xy}{z^2-a^2},$$

$$\left(\frac{du}{dy}\right) = \frac{x^2}{z^2-a^2},$$

$$\left(\frac{du}{dz}\right) = -\frac{2x^2yz}{(z^2-a^2)^2};$$

$$\therefore\ \text{D}u = \frac{x(z^2-a^2)(2y\,dx + x\,dy) - 2x^2yz\,dz}{(z^2-a^2)^2}.$$

50.] If the Variables are so combined in an Equation that it is impossible to express any one explicitly in terms of the others, then

$$\text{F}(x, y, z, \ldots\ldots) = c;$$

and therefore $\text{DF} = 0$, and

$$\left(\frac{d\text{F}}{dx}\right) dx + \left(\frac{d\text{F}}{dy}\right) dy + \left(\frac{d\text{F}}{dz}\right) dz + \ldots\ldots = 0;$$

whence may be determined the absolute variation of any one of them which is due to the variations of all the others; as e.g.

$$dx = -\frac{\left(\frac{d\text{F}}{dy}\right) dy + \left(\frac{d\text{F}}{dz}\right) dz + \ldots\ldots}{\left(\frac{d\text{F}}{dx}\right)}.$$

And similarly may be found the variation of any one of the variables which is due to the variation of any one of the others; as for instance, suppose we have to find the relation between the corresponding variations of x and y in an equation

$$\text{F}(x, y, z, \ldots\ldots) = c,$$

when the other variables do not change value, then

$$\left(\frac{d\text{F}}{dx}\right) dx + \left(\frac{d\text{F}}{dy}\right) dy = 0;$$

$$\therefore \quad \frac{dx}{dy} = -\frac{\left(\frac{d\text{F}}{dy}\right)}{\left(\frac{d\text{F}}{dx}\right)}.$$

Similarly the corresponding variations of x and z may be found from

$$\frac{dx}{dz} = -\frac{\left(\frac{d\text{F}}{dz}\right)}{\left(\frac{d\text{F}}{dx}\right)}.$$

51.] Differentiation of a Function of many Variables, some of which are dependent on the others.

As the principles of differentiation of functions of many variables which have been explained in the five immediately preceding Articles are general, whatever be the values of the variables, *a fortiori* they are applicable to the less general case in which the variables are connected with each other by cer-

tain given relations; and first we will consider the most simple case, viz.:

$$\text{Let} \quad u = \text{F}(x, y) \qquad \text{where} \quad y = f(x);$$

$$\therefore \; \text{D}u = \left(\frac{d\text{F}}{dx}\right) dx + \left(\frac{d\text{F}}{dy}\right) dy \qquad \therefore \; dy = f'(x)\, dx.$$

$$\therefore \; \text{D}u = \left(\frac{d\text{F}}{dx}\right) dx + \left(\frac{d\text{F}}{dy}\right) f'(x)\, dx.$$

$$\therefore \; \frac{\text{D}u}{dx} = \left(\frac{d\text{F}}{dx}\right) + \left(\frac{d\text{F}}{dy}\right) f'(x).$$

The meaning of the several terms of which expression is sufficiently obvious from their symbols.

Ex. 1. $\quad u = \tan^{-1}\dfrac{x}{y} \qquad \text{and} \qquad y = \{a^2 - x^2\}^{\frac{1}{2}};$

$$\therefore \; \text{D}u = \frac{y\,dx - x\,dy}{x^2 + y^2} \qquad dy = -\frac{x\,dx}{\{a^2 - x^2\}^{\frac{1}{2}}}:$$

whence, substituting

$$\text{D}u = \frac{dx}{\{a^2 - x^2\}^{\frac{1}{2}}},$$

a result which is manifestly as it ought to be, if we consider that the relation between u and x, after the elimination of y, is

$$u = \sin^{-1}\frac{x}{a}.$$

Or again, consider the less simple case, $u = \text{F}(x, y, z)$, where

$$y = f(x), \qquad \text{and} \qquad z = \phi(x);$$

$$\therefore \; \text{D}u = \left(\frac{d\text{F}}{dx}\right) dx + \left(\frac{d\text{F}}{dy}\right) dy + \left(\frac{d\text{F}}{dz}\right) dz.$$

$$dy = f'(x)\, dx,$$

$$dz = \phi'(x)\, dx;$$

whence, substituting and dividing by dx,

$$\frac{\text{D}u}{dx} = \left(\frac{d\text{F}}{dx}\right) + \left(\frac{d\text{F}}{dy}\right) f'(x) + \left(\frac{d\text{F}}{dz}\right) \phi'(x).$$

Ex. 2. $\quad u = \sin ax + \sin by + \tan^{-1}\dfrac{y}{z},$

$$y = \sec x,$$

$$z = \operatorname{cosec} x,$$

$$\therefore \quad \mathrm{D}u = a\cos ax\,dx + b\cos by\,dy + \frac{zdy - ydz}{y^2 + z^2}.$$

$$dy = \sec x \tan x\, dx,$$

$$dz = -\operatorname{cosec} x \cot x\, dx;$$

$$\therefore \quad \mathrm{D}u = a\cos ax\,dx + b\cos(b\sec x)\sec x\tan x\,dx + dx.$$

Or again, suppose $u = \mathrm{F}(x, y)$,

and $x = f(t)$ $y = \phi(t)$,

t being another variable.

$$\therefore \quad \mathrm{D}u = \left(\frac{d\mathrm{F}}{dx}\right) dx + \left(\frac{d\mathrm{F}}{dy}\right) dy,$$

$$= \left(\frac{d\mathrm{F}}{dx}\right) f'(t)\,dt + \left(\frac{d\mathrm{F}}{dy}\right) \phi'(t)\,dt,$$

$$\therefore \quad \frac{\mathrm{D}u}{dt} = \left(\frac{d\mathrm{F}}{dx}\right) f'(t) + \left(\frac{d\mathrm{F}}{dy}\right) \phi'(t).$$

Or again, suppose $u = \mathrm{F}(x, y, z)$,

and $z = f(x, y)$;

$$\therefore \quad \mathrm{D}u = \left(\frac{d\mathrm{F}}{dx}\right) dx + \left(\frac{d\mathrm{F}}{dy}\right) dy + \left(\frac{d\mathrm{F}}{dz}\right) dz,$$

but $\mathrm{D}z$ (or dz) $= \left(\frac{df}{dx}\right) dx + \left(\frac{df}{dy}\right) dy,$

$$\therefore \quad \mathrm{D}u = \left\{\left(\frac{d\mathrm{F}}{dx}\right) + \left(\frac{d\mathrm{F}}{dz}\right)\left(\frac{df}{dx}\right)\right\} dx + \left\{\left(\frac{d\mathrm{F}}{dy}\right) + \left(\frac{d\mathrm{F}}{dz}\right)\left(\frac{df}{dy}\right)\right\} dy.$$

Ex. 3. $u = x^{yz}$ and $z = \sin(xy)$;

$$\therefore \quad \mathrm{D}u = yz\,x^{yz-1}\,dx + x^{yz}\log_e x\,(y\,dz + z\,dy),$$

$$\mathrm{D}z \text{ (or } dz) = \cos(xy)\,(y\,dx + x\,dy);$$

$$\therefore \quad \mathrm{D}u = y\,x^{yz}\left\{\frac{z}{x} + y\log_e x\cos(xy)\right\} dx$$
$$+ \log_e x\; x^{yz}\,\{z + xy\cos(xy)\}\,dy.$$

Ex. 4. Given $z = \mathrm{F}\{y + xf(z)\}$, where y and x are independent variables, it is required to find $\left(\frac{dz}{dx}\right)$ and $\left(\frac{dz}{dy}\right)$.

or the sake of abbreviation symbolizing $F'\{y+xf(z)\}$ by F', have

$$\left(\frac{dz}{dx}\right) = \left\{ f(z) + xf'(z)\left(\frac{dz}{dx}\right)\right\} F',$$

$$\left(\frac{dz}{dy}\right) = \left\{ 1 + xf'(z)\left(\frac{dz}{dy}\right)\right\} F':$$

$$\therefore \quad \left(\frac{dz}{dx}\right) = \frac{f(z)\,F'}{1-xf'(z)\,F'};$$

$$\left(\frac{dz}{dy}\right) = \frac{F'}{1-xf'(z)\,F'},$$

$$\therefore \quad \left(\frac{dz}{dx}\right) = f(z)\left(\frac{dz}{dy}\right);$$

h result is of great importance in the solution of a future lem. See Art. 84.

he cases of differentiation of functions of the preceding kind so various and numerous, that it is impossible to discuss all em; but the principles explained and illustrated as above applicable universally, and from the examples given the ent ought to find no difficulty in solving other similar ones.

CHAPTER III.

SUCCESSIVE DIFFERENTIATION, AND THEOREMS DEPENDENT ON IT.

SECTION I.

ON SUCCESSIVE DIFFERENTIATION OF AN EXPLICIT FUNCTION OF ONE VARIABLE.

52.] In the former part of the last Chapter rules were constructed for differentiating explicit functions of one variable, and thus of deducing from $f(x)$, which was assumed as the typical form, $d.f(x)$ or $f'(x)\,dx$; in this case x was made to increase or grow by the infinitesimal dx. Suppose now that x is made to increase again by an infinitesimal variation, it is our object to inquire what effect such a second increase will have on $f(x)$ and on $f'(x)\,dx$; this second increment may or may not be equal to the former one: doubtless the simpler case will be when it is equal, and therefore we will consider it first with this limitation, and subsequently discuss the general case when the successive increments of x are not equal. When x increases by equal increments, or *grows*, as we may say, at an uniform rate, it is said to be an *equicrescent variable.*

Or we may consider the subject from another point of view; $f'(x)$ is in general a new function of x; and therefore as it was derived from $f(x)$, so by a similar process may another function be derived from it. This new function by an analogous notation is symbolized by $f''(x)$, and will in general be another function of x, and thus will admit of having another function derived from it by a similar process, which will be symbolized by $f'''(x)$, and so on. Thus may *derivation* be considered an algebraical artifice by which successive functions are formed, each from the preceding one, and are called the derived-functions or *derivatives* of different orders of $f(x)$, viz.:

$$f(x),\ f'(x),\ f''(x),\ f'''(x),\ \ldots\ldots f^n(x),\ f^{n+1}(x); \qquad (1)$$

$f(x)$ is called the primitive function; $f'(x)$ the first-derived; $f''(x)$ the second-derived; $f^n(x)$ the nth derived-function of $f(x)$. It is also to be observed, that each function in the above series is the first-derived of the immediately preceding function.

In the same way therefore as $d.f(x) = f'(x)\,dx$, so does $d.f'(x) = f''(x)\,dx$, and so on; whence we have the following series of equations:

$$\left.\begin{aligned} d.f(x) &= f'(x)\,dx, \\ d.f'(x) &= f''(x)\,dx, \\ d.f''(x) &= f'''(x)\,dx, \\ &\cdots\cdots \\ d.f^{n-1}(x) &= f^n(x)\,dx. \end{aligned}\right\} \qquad (2)$$

The dx s, which are factors on the right-hand side of the above equations, are the several increments of x which give rise to the differentials of $f(x)$, $f'(x)$, ... and therefore are not necessarily all equal; but, as above, to consider the simpler case, let them be equal, so that the several functions vary by reason of *equal* variations of their variables: that is, let x be equicrescent; then, since

$$y = f(x),$$
$$dy = d.f(x) = f'(x)\,dx.$$

And since $f'(x)$ is a function of x, we may differentiate again; whence, as dx is constant, we have

$$d.dy = d.f'(x)\,dx;$$

and since $d.dy$ signifies, that the operation symbolized by the character d is to be performed twice on y, and one operation on the back of the other, we may, in accordance with the notation of the index law, abbreviate $d.dy$ into d^2y; and replacing $d.f'(x)$ by $f''(x)\,dx$, and writing dx^2 for the square of dx, since the dx s are equal, we have

$$d^2y = f''(x)\,dx^2.$$

Similarly differentiating again, and writing d^3y for $d.d^2y$, and dx^3 for the cube of dx, we have

$$d^3y = d.f''(x)\, dx^2$$
$$= f'''(x)\, dx^3,$$

and so on; whence the general expression becomes

$$d^n y = f^n(x)\, dx^n.$$

The quantities dy, d^2y, d^3y, ... $d^n y$, ... are severally called the first, second, third, ... nth ...... differentials of y; hence it follows, that the differential of the nth order of y or $f(x)$ is equal to the product of the nth derived-function and $(dx)^n$.

Also $f''(x)$, $f'''(x)$, ... $f^n(x)$ ...... are severally called the second, third, ... nth ...... differential coefficients, in accordance with the principle of nomenclature given in Art. 18; because in the above equations they are the coefficients of dx^2, dx^3, ... dx^n ......

Hence also we have,

$$\left.\begin{aligned} y &= f(x), \\ \frac{dy}{dx} &= f'(x), \\ \frac{d^2y}{dx^2} &= f''(x), \\ &\cdots\cdots \\ \frac{d^n y}{dx^n} &= f^n(x). \end{aligned}\right\} \qquad (3)$$

By which form of writing the fractions on the left-hand side of the equations, it is indicated that x is *equicrescent*. Also let the difference of notation be observed between $d^n y$ and dx^n.

Hence also it appears, that the nth derived-function of $f(x)$ is equal to the ratio of the nth differential of the function to the nth power of dx: dx being the quantity by which x has varied in each of the successive derivations; hence also it is plain that, if $f^n(x)$ be finite, $d^n y$ is an infinitesimal of the nth order with respect to dx as a base.

It is also to be observed, that as $\frac{dy}{dx}$ represents the ratio of the variation of y, or $f(x)$ to that of x, so $\frac{d^2y}{dx^2} = f''(x) = \frac{d.f'(x)}{dx}$ represents the ratio of the variation of $f'(x)$, that is of $\frac{dy}{dx}$, to

he variation of x; and so $\frac{d^3y}{dx^3}$ represents the ratio of the ariation of $\frac{d^2y}{dx^2}$ to that of x; and similarly of the other erived-functions.

Ex. 1.
$$y = x^n,$$
$$\frac{dy}{dx} = nx^{n-1},$$
$$\frac{d^2y}{dx^2} = n(n-1)\, x^{n-2},$$
$$\frac{d^3y}{dx^3} = n(n-1)\,(n-2)\, x^{n-3},$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\frac{d^ry}{dx^r} = n(n-1)\,(n-2) \ldots (n-r+2)\,(n-r+1)\, x^{n-r}.$$

Ex. 2.
$$y = \log_e x,$$
$$\frac{dy}{dx} = \frac{1}{x} = x^{-1},$$
$$\frac{d^2y}{dx^2} = -\, x^{-2},$$
$$\frac{d^3y}{dx^3} = (-)^2\, 1.2.\, x^{-3},$$
$$\frac{d^4y}{dx^4} = (-)^3\, 1.2.3.\, x^{-4},$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\frac{d^ry}{dx^r} = (-)^{r-1}\, 1.2.3. \ldots\ldots (r-2)\,(r-1)\, x^{-r}.$$

Ex. 3.
$$y = a^x,$$
$$\frac{dy}{dx} = \log_e a\; a^x,$$
$$\frac{d^2y}{dx^2} = (\log_e a)^2\, a^x,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\frac{d^ry}{dx^r} = (\log_e a)^r\, a^x.$$

Ex. 4. $y = \sin x,$

$$\frac{dy}{dx} = \cos x = \sin\left(x + \frac{\pi}{2}\right),$$

$$\frac{d^2y}{dx^2} = \cos\left(x + \frac{\pi}{2}\right) = \sin\left(x + 2\frac{\pi}{2}\right),$$

. . . .

$$\frac{d^ry}{dx^r} = \sin\left(x + r\frac{\pi}{2}\right).$$

It appears therefore that the effect of one derivation on $\sin x$ is to increase the arc by $\frac{\pi}{2}$, and therefore it might at once be concluded, that the effect of r derivations is to increase the arc by $r\frac{\pi}{2}$.

Similarly, if

$$y = \cos x,$$

$$\frac{d^ry}{dx^r} = \cos\left(x + r\frac{\pi}{2}\right).$$

Ex. 5. $y = \frac{1}{x^2 - a^2} = \frac{1}{2a}\left\{\frac{1}{x-a} - \frac{1}{x+a}\right\},$

$$= \frac{1}{2a}\left\{(x-a)^{-1} - (x+a)^{-1}\right\};$$

$$\therefore \frac{dy}{dx} = -\frac{1}{2a}\left\{(x-a)^{-2} - (x+a)^{-2}\right\}.$$

$$\frac{d^2y}{dx^2} = (-)^2\frac{1.2}{2a}\left\{(x-a)^{-3} - (x+a)^{-3}\right\},$$

$$\frac{d^3y}{dx^3} = (-)^3\frac{1.2.3}{2a}\left\{(x-a)^{-4} - (x+a)^{-4}\right\},$$

. . . .

$$\frac{d^ry}{dx^r} = (-)^r\frac{1.2.3\,..r}{2a}\left\{(x-a)^{-(r+1)} - (x+a)^{-(r+1)}\right\}.$$

A similar method of decomposition into partial fractions may also be employed to find the general derived-function of $\frac{x}{x^2-a^2}$, $\frac{1}{x^2+a^2}$, $\frac{x}{x^2+a^2}$, and other functions of a similar form.

Ex. 6. $y = e^{ax}\sin nx,$

$$\frac{dy}{dx} = e^{ax}\{a\sin nx + n\cos nx\}.$$

Let $\left\{\begin{array}{l} a = k\cos\phi \\ n = k\sin\phi \end{array}\right\} \quad \therefore \quad \left\{\begin{array}{l} k^2 = a^2+n^2 \\ \tan\phi = \dfrac{n}{a}; \end{array}\right.$

on the legitimacy of which substitutions, see Art. 57.

$$\therefore \quad \frac{dy}{dx} = k\,e^{ax}\sin(nx+\phi);$$

whence it is manifest, that

$$\frac{d^ry}{dx^r} = k^r\,e^{ax}\sin(nx+r\phi),$$

$$= (a^2+n^2)^{\frac{r}{2}}\,e^{ax}\sin(nx+r\phi).$$

53.] The following theorem, due to Leibnitz, and commonly called Leibnitz's Theorem, may be conveniently employed to find the general (the rth) derived-function of a product of two functions of x.

Let $f(x)$ be the function of x, of which let the two factors be u and v, so that $\quad y = f(x) = u\times v;$

$$dy = v\,du + dv\,u,$$

$$d^2y = v\,d^2u + 2\,dv\,du + d^2v\,u,$$

$$d^3y = v\,d^3u + 3\,dv\,d^2u + 3\,d^2v\,du + d^3v\,u;$$

the law of the coefficients being, as it is manifest, that of the coefficients of $(1+x)^r$; and, to shew that such is always the case, let us assume that

$$d^{r-1}y = v d^{r-1}u + \frac{r-1}{1}\,dv\,d^{r-2}u + \frac{(r-1)\,(r-2)}{1.2}\,d^2v\,d^{r-3}u$$
$$+ \ldots\ldots + \frac{r-1}{1}\,du\,d^{r-2}v + u d^{r-1}v,$$

and differentiate; whereby

$$d^ry = v\,d^ru + \frac{r}{1}\,dv\,d^{r-1}u + \frac{r\,(r-1)}{1.2}\,d^2v\,d^{r-2}u$$
$$+ \ldots\ldots + r\,du\,d^{r-1}v + u\,d^rv. \qquad (4)$$

Therefore, if the formula be true for $(r-1)$ it is true for r; but

it is true for 2, $\therefore$ it is true for 3; and therefore it is true for *all positive and integral* values of r*.

$$\therefore \quad d^r(u \times v) = f^r(x)\, dx^r = v\, d^r u + \frac{r}{1}\, dv\, d^{r-1}u$$

$$+ \frac{r(r-1)}{1.2}\, d^2 v\, d^{r-2}u + \ldots\ldots \qquad (5)$$

$$\therefore \quad f^r(x) = v \frac{d^r u}{dx^r} + \frac{r}{1}\frac{dv}{dx}\frac{d^{r-1}u}{dx^{r-1}} + \frac{r(r-1)}{1.2}\frac{d^2 v}{dx^2}\frac{d^{r-2}u}{dx^{r-2}} + \ldots \quad (6)$$

In applying the last formula to a given example, let the student be careful to assume as u that function whose general derived-function is the more easily calculated.

Ex. 1. To find the rth derived-function of $e^{ax}\, x^n$.

$$u = e^{ax}, \qquad\qquad v = x^n,$$

$$\frac{du}{dx} = a\, e^{ax}, \qquad\qquad \frac{dv}{dx} = n\, x^{n-1},$$

$$\frac{d^2 u}{dx^2} = a^2\, e^{ax}, \qquad\qquad \frac{d^2 v}{dx^2} = n(n-1)\, x^{n-2},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \qquad\qquad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\frac{d^r u}{dx^r} = a^r\, e^{ax};$$

$$\therefore \quad \frac{d^r(e^{ax}\, x^n)}{dx^r} = a^r\, e^{ax}\, x^n + \frac{r}{1}\, a^{r-1}\, e^{ax}\, n x^{n-1}$$

$$+ \frac{r(r-1)}{1.2}\, a^{r-2}\, e^{ax}\, n(n-1)\, x^{n-2} + \ldots\ldots$$

$$= e^{ax}\left\{a^r\, x^n + \frac{r}{1}\, a^{r-1}\, n\, x^{n-1}\right.$$

$$\left. + \frac{r(r-1)}{1.2}\, a^{r-2}\, n(n-1)\, x^{n-2} + \ldots\ldots\right\}$$

Ex. 2. To find the rth derived-function of $e^{ax}\cos nx\, x^m$.

$$u = e^{ax}\cos nx, \qquad\qquad v = x^m;$$

$\therefore$ by Ex. 6, Art. 52, if

* On this mode of inductively proving the above theorem, see some remarks in a "System of Logic," by John Stuart Mill, book III, ch. ii, §. 2, 2nd edition, London, 1846.

$$\tan\phi = \frac{n}{a} \text{ and } k = \{n^2+a^2\}^{\frac{1}{2}},$$

$$\frac{du}{dx} = k\, e^{ax} \cos(nx+\phi), \qquad \frac{dv}{dx} = mx^{m-1},$$

$$\frac{d^2u}{dx^2} = k^2\, e^{ax} \cos(nx+2\phi), \qquad \frac{d^2v}{dx^2} = m(m-1)\, x^{m-2},$$

$$\cdots\cdots\cdots\cdots \qquad \cdots\cdots\cdots\cdots$$

$$\frac{d^ru}{dx^r} = k^r\, e^{ax} \cos(nx+r\phi).$$

$$\begin{aligned} \therefore\quad \frac{d^r(e^{ax}\cos nx\, x^m)}{dx^r} = e^{ax}\, \{ & k^r \cos(nx+r\phi)\, x^m \\ & + \frac{r}{1} k^{r-1} \cos(nx+(r-1)\phi)\, mx^{m-1} \\ & + \frac{r(r-1)}{1.2} k^{r-2} \cos(nx+(r-2)\phi)\, m(m-1)\, x^{m-2} \\ & + \cdots\cdots\cdots\cdots\cdots\cdots \} \end{aligned}$$

SECTION 2.—Maclaurin's Theorem for the Expansion of an Explicit Function of One Variable.

54.] Suppose $f(x)$ to be a function of x, and to be capable of expansion in a series of ascending powers of x, of the form

$$f(x) = A_0 + A_1 x + A_2 x^2 + \cdots\cdots + A_n x^n + \cdots\cdots \tag{7}$$

and which is subject to the following conditions:

1st. $A_0, A_1, A_2 \cdots\cdots A_n \cdots\cdots$ are constants, and to be determined.

2nd. $A_0, A_1, A_2 \cdots\cdots A_n \cdots\cdots$ do not become infinite for any value of x for which the series is applied.

3rd. The series contains no terms involving negative or fractional powers of x.

Then, since

$$\begin{aligned} f(x) &= A_0 + A_1 x + A_2 x^2 + \cdots\cdots + A_n x^n + \cdots\cdots \\ \therefore\ f'(x) &= A_1 + 2A_2 x + 3A_3 x^2 + \cdots\cdots \\ f''(x) &= 2A_2 + 2.3A_3 x + \cdots\cdots \\ f'''(x) &= 2.3A_3 + \cdots\cdots \\ &\cdots\cdots\cdots\cdots \\ f^n(x) &= 1.2.3 \cdots\cdots (n-1)\, n\, A_n + \cdots\cdots \end{aligned}$$

In these equations let $x = 0$; then, since by the conditions (2) and (3) none of the quantities assume the indeterminate form $\frac{0}{0}$, or become infinite, we have

$$
\begin{aligned}
f(0) &= A_0, \\
f'(0) &= A_1, \\
f''(0) &= 1.2.A_2, & \therefore\quad A_2 &= \frac{f''(0)}{1.2}, \\
f'''(0) &= 1.2.3.A_3 & \therefore\quad A_3 &= \frac{f'''(0)}{1.2.3}; \\
&\cdots\cdots & &\cdots\cdots \\
f^n(0) &= 1.2.3\ldots\ldots n\,A_n & \therefore\quad A_n &= \frac{f^n(0)}{1.2.3\ldots n}.
\end{aligned}
$$

Whence, substituting in equation (7), we have

$$f(x) = f(0) + f'(0)\frac{x}{1} + f''(0)\frac{x^2}{1.2} + \ldots + f^n(0)\frac{x^n}{1.2.3\ldots n} + \ldots (8)$$

$f(0), f'(0), f''(0) \ldots\ldots$ being the values of $f(x), f'(x), f''(x), \ldots\ldots$ when $x = 0$.

The series was discovered by Stirling, an English mathematician of the early part of the last century; but having been introduced by Maclaurin into his Treatise of Fluxions, has been generally called by his name.

55.] For certain functions of x, such as those of the form $(a+x)^n$, where n is positive and integral, the derived-functions will vanish after a certain number of differentiations, and therefore the number of terms of the above series is limited in such cases; but generally the successively derived-functions are functions of x, and the series is continued to an infinite number of terms; but the sum of all the terms after the nth may be expressed as follows, by an algebraical formula. Since

$$
\begin{aligned}
f(x) = f(0) + f'(0)\frac{x}{1} + f''(0)\frac{x^2}{1.2} + \ldots + f^{n-1}(0)\frac{x^{n-1}}{1.2.3\ldots(n-1)} \\
+ f^n(0)\frac{x^n}{1.2.3\ldots n} + f^{n+1}(0)\frac{x^{n+1}}{1.2.3\ldots(n+1)} \\
+ f^{n+2}(0)\frac{x^{n+2}}{1.2.3\ldots(n+2)} + \ldots\ldots \qquad (9)
\end{aligned}
$$

The sum of all the terms after the nth

$$= f^n(0)\frac{x^n}{1.2.3\dots n} + f^{n+1}(0)\frac{x^{n+1}}{1.2.3\dots(n+1)} + \dots\dots$$

$$= \frac{x^n}{1.2.3\dots n}\{f^n(0) + f^{n+1}(0)\frac{x}{n+1}$$

$$+ f^{n+2}(0)\frac{x^2}{(n+1)(n+2)} + \dots\dots\} \qquad (10)$$

$$= \frac{x^n}{1.2.3\dots n}\{\text{a quantity} > f^n(0) \text{ and } < f^n(x)\} \qquad (11)$$

the latter factor of (10) is, I say, greater than $f^n(0)$, because the sum of the series is (algebraically) greater than its first term; and is less than $f^n(x)$, because by reason of (9)

$$f^n(x) = f^n(0) + f^{n+1}(0)\frac{x}{1} + f^{n+2}(0)\frac{x^2}{1.2} + \dots\dots$$

And with the exception of the first term of this series, each term of it is greater than the corresponding term of the series given in (10), inasmuch as the denominators are severally less. Hence representing by θ *some* positive and proper fraction, we may symbolize (11) by

$$\frac{x^n}{1.2.3\dots n}f^n(\theta x); \qquad (12)$$

for the latter factor is too small when $\theta = 0$, and is too large when $\theta = 1$.

Hence we may write the series as follows:

$$f(x) = f(0) + f'(0)\frac{x}{1} + f''(0)\frac{x^2}{1.2} + \dots + \frac{x^n}{1.2.3\dots n}f^n(\theta x) \quad (13)$$

which has a definite number of terms; and therefore the only apparent difference in the absolute equality of the two sides of the equation is that which arises from θ being an undetermined fraction greater than zero and less than unity.

As this series however is of great importance in the application of the Calculus, the proof of it must not rest on any fallacious assumption, or any vague limitations which may be too wide or too narrow, such as those of the stated conditions. Hence arises the need of a rigorous and exact proof of it, and of one which will limit the extent of its applicability, and which will be given hereafter; and therefore the explanation of the last two articles is to be considered only as yielding a presumption that such a series as (13) is likely to be true.

56.] **Examples of Maclaurin's Theorem.**

Ex. 1. Let

$$f(x) = (a+x)^n, \qquad \therefore \quad f(0) = a^n,$$
$$f'(x) = n(a+x)^{n-1}, \qquad f'(0) = na^{n-1},$$
$$f''(x) = n(n-1)(a+x)^{n-2}, \qquad f''(0) = n(n-1)a^{n-2},$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \qquad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

whence, if n be positive and integral,

$$f^n(x) = n(n-1)(n-2)\ldots 3.2.1, \quad f^n(0) = n(n-1)(n-2)\ldots 3.2.1;$$

$$\therefore \quad f(x) = (a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1.2}a^{n-2}x^2 + \ldots\ldots + nax^{n-1} + x^n, \tag{14}$$

the common Binomial Theorem, which is therefore a particular case of Maclaurin's Theorem.

Ex. 2.

$$f(x) = e^x, \qquad \therefore \quad f(0) = 1,$$
$$f'(x) = e^x, \qquad f'(0) = 1,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \qquad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$f^n(x) = e^x, \qquad f^n(\theta x) = e^{\theta x};$$

$$\therefore \quad e^x = 1 + \frac{x}{1} + \frac{x^2}{1.2} + \cdots + \frac{x^{n-1}}{1.2.3\ldots(n-1)} + \frac{x^n}{1.2.3\ldots n}e^{\theta x}. \tag{15}$$

Ex. 3.

$$f(x) = \sin x, \qquad f(0) = 0,$$
$$f'(x) = \sin\left(x + \frac{\pi}{2}\right), \qquad f'(0) = 1,$$
$$f''(x) = \sin\left(x + 2\frac{\pi}{2}\right), \qquad f''(0) = 0,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \qquad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$f^n(x) = \sin\left(x + n\frac{\pi}{2}\right), \qquad f^n(\theta x) = \sin\left(\theta x + n\frac{\pi}{2}\right).$$

$$\therefore \quad \sin x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \ldots\ldots$$
$$+ \frac{x^{n-1}}{1.2.3\ldots(n-1)}\sin\left\{(n-1)\frac{\pi}{2}\right\} + \frac{x^n}{1.2.3\ldots n}\sin\left(\theta x + n\frac{\pi}{2}\right) \tag{16}$$

Ex. 4.

$$f(x) = \cos x, \qquad \therefore \; f(0) = 1,$$

$$f'(x) = \cos\left(n + \frac{\pi}{2}\right), \qquad f'(0) = 0,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \qquad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$f^n(x) = \cos\left(x + n\frac{\pi}{2}\right), \qquad f^n(\theta x) = \cos\left(\theta x + n\frac{\pi}{2}\right);$$

$$\therefore \quad \cos x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \ldots\ldots$$

$$+ \frac{x^{n-1}}{1.2.3\ldots(n-1)} \cos\left\{(n-1)\frac{\pi}{2}\right\} + \frac{x^n}{1.2.3\ldots n}\cos\left(\theta x + n\frac{\pi}{2}\right) \quad (17)$$

Ex. 5.

$$f(x) = \log_e (1+x), \qquad \therefore \; f(0) = 0,$$

$$f'(x) = (1+x)^{-1}, \qquad f'(0) = 1,$$

$$f''(x) = -(1+x)^{-2}, \qquad f''(0) = -1,$$

$$f'''(x) = (-)^2\, 1.2\, (1+x)^{-3}, \qquad f'''(0) = 1.2,$$

$$f''''(x) = (-)^3\, 1.2.3\, (1+x)^{-4}, \qquad f''''(0) = -1.2.3,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \qquad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$f^n(x) = (-)^{n-1}\, 1.2.3 \ldots (n-1)\, (1+x)^{-n},$$

$$f^n(\theta x) = (-)^{n-1}\, 1.2.3 \ldots (n-1)\, (1+\theta x)^{-n}$$

$$\therefore \quad \log_e (1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots\ldots$$

$$(-)^{n-2}\frac{x^{n-1}}{n-1} \; (-)^{n-1}\frac{1}{n}\left(\frac{x}{1+\theta x}\right)^n \ldots\ldots \qquad (18)$$

Ex. 6.

$$f(x) = \sin^{-1} x, \qquad f(0) = 0,$$

$$f'(x) = \frac{1}{(1-x^2)^{\frac{1}{2}}} = (1-x^2)^{-\frac{1}{2}}, \qquad f'(0) = 1,$$

$$f''(x) = x\,(1-x^2)^{-\frac{3}{2}}, \qquad f''(0) = 0,$$

$$f'''(x) = (1-x^2)^{-\frac{3}{2}} + 3\,x^2\,(1-x^2)^{-\frac{5}{2}}, \qquad f'''(0) = 1,$$

$$f''''(x) = 9x\,(1-x^2)^{-\frac{5}{2}} - 3.5\,x^3\,(1-x^2)^{-\frac{7}{2}}, \qquad f''''(0) = 0,$$

$$f'''''(x) = 9\,(1-x^2)^{-\frac{5}{2}} + \ldots\ldots \qquad f'''''(0) = 9.$$

The general derived-function is of too complicated a form to be useful;

$$\therefore \quad \sin^{-1} x = x + \frac{x^3}{1.2.3} + \frac{9x^5}{1.2.3.4.5} + \ldots\ldots \quad (19)$$

But $\sin^{-1} x$ may be more easily expanded by the following artifice, which adapts itself to those functions of which a derived-function assumes an algebraical form.

Differentiating equation (8), Art. 54, we have

$$f'(x) = f'(0) + f''(0)\frac{x}{1} + f'''(0)\frac{x^2}{1.2} + f''''(0)\frac{x^3}{1.2.3} + \ldots\ldots$$

$$+ f^{n+1}(0)\frac{x^n}{1.2.3\ldots n} + \ldots\ldots \quad (20)$$

and from above,

$$f'(x) = (1-x^2)^{-\frac{1}{2}},$$

$$= 1 + \frac{x^2}{2} + \frac{1.3}{2^2}\frac{x^4}{1.2} + \frac{1.3.5}{2^3}\frac{x^6}{1.2.3} + \ldots\ldots$$

$$+ \frac{1.3.5.7\ldots(2r-1)}{2^r}\frac{x^{2r}}{1.2.3\ldots r} + \ldots\ldots \quad (21)$$

whence, equating coefficients of the same power of x in the two series, we have

$$f'(0) = 1, \quad f''(0) = 0, \quad f'''(0) = \frac{1.1.2}{2.1}, \quad f''''(0) = 0,$$

$$f'''''(0) = \frac{1.3}{2^2}\frac{1.2.3.4}{1.2},$$

$$f^{2r+1}(0) = \frac{1.3.5.7\ldots(2r-1)\,1.2.3\ldots(2r-1)\,2r}{2^r\,1.2.3\ldots(r-1)\,r};$$

whence we have

$$\sin^{-1} x = x + \frac{1}{2}\frac{1.2}{1}\frac{x^3}{1.2.3} + \frac{1.3}{2^2}\frac{1.2.3.4}{1.2}\frac{x^5}{1.2.3.4.5} + \ldots\ldots$$

$$+ \frac{1.3.5\ldots(2r-1)}{2^r}\frac{1.2.3.4\ldots(2r-1)2r}{1.2.3\ldots r}\frac{x^{2r+1}}{1.2.3\ldots 2r(2r+1)} + \ldots$$

$$\therefore \quad \sin^{-1} x = x + \frac{x^3}{6} + \frac{9x^5}{1.2.3.4.5} + \ldots\ldots$$

$$+ \frac{1.3.5\ldots(2r-1)\,x^{2r+1}}{2.4.6\ldots 2r\,(2r+1)} + \ldots\ldots \quad (22)$$

$\cos^{-1} x$, $\tan^{-1} x$, $\text{cotan}^{-1} x$, $\log(1+x)$, are other functions which may be conveniently expanded by this method.

57.] As many properties of some series which have been expanded in the last Article will be necessary in the sequel of the Treatise, it is most convenient to introduce them here, though they may more properly be considered to belong to analytical trigonometry.

By an imaginary or impossible quantity is meant, one of the form

$$a + b\sqrt{-1}; \tag{23}$$

a and b being symbols of positive or negative possible quantities, and the symbol $\sqrt{-1}$ being that, which, when squared, is equal to -1: two such expressions, which differ only in the sign of $\sqrt{-1}$, are said to be *conjugate* to each other. Thus

$$a + b\sqrt{-1} \text{ and } a - b\sqrt{-1}$$

are called conjugate imaginary expressions; and it is to be observed, that the product of two such conjugate expressions, viz.

$$(a + b\sqrt{-1})\,(a - b\sqrt{-1}) = a^2 + b^2. \tag{24}$$

Now such an expression as (23) may always be put under the form

$$r(\cos\theta + \sqrt{-1}\sin\theta); \tag{25}$$

in which case r is called the *modulus* of the expression $a + b\sqrt{-1}$. For let

$$\left.\begin{aligned} a &= r\cos\theta, \\ b &= r\sin\theta, \end{aligned}\right\} \qquad \left.\begin{aligned} \therefore\quad r^2 &= a^2+b^2, \\ \tan\theta &= \frac{b}{a}; \end{aligned}\right\} \tag{26}$$

and as a and b are possible quantities, a^2+b^2 is a positive quantity, and therefore r is possible; and as $\tan\theta$ passes through all values, from $-\infty$ through 0 to $+\infty$, as θ increases from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$, whatever be the relative signs and magnitudes of a and b, there is always some arc between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ which will satisfy the equation $\tan\theta = \frac{b}{a}$; therefore the substitutions made for a and b are possible, and therefore $a + b\sqrt{-1}$ may always be put in the form $r(\cos\theta + \sqrt{-1}\sin\theta)$. Also,

$$\frac{\sin\theta}{b} = \frac{\cos\theta}{a} = \frac{1}{\{a^2+b^2\}^{\frac{1}{2}}} \text{ by Preliminary Theorem I.}$$

58.] In series (15), Art. 56, successively write for x, $x\sqrt{-1}$ and $-x\sqrt{-1}$; then

$$e^{x\sqrt{-1}} = 1 + x\sqrt{-1} - \frac{x^2}{1.2} - \sqrt{-1}\frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \dots \quad (27)$$

$$e^{-x\sqrt{-1}} = 1 - x\sqrt{-1} - \frac{x^2}{1.2} + \sqrt{-1}\frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} - \dots \quad (28)$$

$$\therefore\ e^{x\sqrt{-1}} + e^{-x\sqrt{-1}} = 2\left(1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \dots\dots\right)$$

$$= 2\cos x, \text{ by series (17), Art. 56;} \quad (29)$$

$$e^{x\sqrt{-1}} - e^{-x\sqrt{-1}} = 2\sqrt{-1}\left(x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \dots\right)$$

$$= 2\sqrt{-1}\sin x, \text{ by series (16), Art. 56;} \quad (30)$$

$$\therefore\ e^{x\sqrt{-1}} = \cos x + \sqrt{-1}\sin x; \quad (31)$$

$$e^{-x\sqrt{-1}} = \cos x - \sqrt{-1}\sin x; \quad (32)$$

whence, by division,

$$e^{2x\sqrt{-1}} = \frac{\cos x + \sqrt{-1}\sin x}{\cos x - \sqrt{-1}\sin x} = \frac{1 + \sqrt{-1}\tan x}{1 - \sqrt{-1}\tan x}; \quad (33)$$

and taking the Napierian Logarithms of both sides of the equation,

$$2x\sqrt{-1} = \log(1 + \sqrt{-1}\tan x) - \log(1 - \sqrt{-1}\tan x),$$

$$= \sqrt{-1}\tan x + \frac{(\tan x)^2}{2} - \sqrt{-1}\frac{(\tan x)^3}{3}$$

$$- \frac{(\tan x)^4}{4} + \sqrt{-1}\frac{(\tan x)^5}{5} + \dots\dots$$

$$- \left\{ -\sqrt{-1}\tan x + \frac{(\tan x)^2}{2} + \sqrt{-1}\frac{(\tan x)^3}{3}\right.$$

$$\left. - \frac{(\tan x)^4}{4} - \sqrt{-1}\frac{(\tan x)^5}{5} + \dots\dots \right\}$$

the series being expanded by equation (18), Art. 56; whence, equating impossible parts, and dividing both sides by $2\sqrt{-1}$, we have

$$x = \tan x - \frac{(\tan x)^3}{3} + \frac{(\tan x)^5}{5} - \dots\dots \quad (34)$$

which is a series useful for the calculation of π.

Again, by equation (31),

$$e^{x\sqrt{-1}} = \cos x + \sqrt{-1}\sin x,$$

$$e^{y\sqrt{-1}} = \cos y + \sqrt{-1}\sin y;$$

$\therefore$ by multiplication $e^{(x+y)\sqrt{-1}} = \cos x \cos y - \sin x \sin y$

$$+\sqrt{-1}(\cos x \sin y + \cos y \sin x), \qquad (35)$$

but $\quad e^{(x+y)\sqrt{-1}} = \cos(x+y) + \sqrt{-1}\sin(x+y); \qquad (36)$

wherefore, equating the possible and impossible parts of the equal quantities (35) and (36), we have the fundamental trigonometrical formulæ,

$$\cos(x+y) = \cos x \cos y - \sin x \sin y, \qquad (37)$$

$$\sin(x+y) = \sin x \cos y + \cos x \sin y. \qquad (38)$$

Again:

$$(\cos x + \sqrt{-1}\sin x)(\cos y + \sqrt{-1}\sin y)(\cos z + \sqrt{-1}\sin z)\ldots$$

$$= e^{x\sqrt{-1}} \times e^{y\sqrt{-1}} \times e^{z\sqrt{-1}} \ldots\ldots = e^{(x+y+z+\ldots)\sqrt{-1}},$$

$$= \cos(x+y+z+\ldots) + \sqrt{-1}\sin(x+y+z+\ldots)\ldots\ldots \qquad (39)$$

whence, if $x = y = z = \ldots\ldots$ to m quantities,

$$(\cos x + \sqrt{-1}\sin x)^m = \cos mx + \sqrt{-1}\sin mx \ldots\ldots \qquad (40)$$

which is De Moivre's Theorem.

By these processes therefore the multiplication of a series of factors of the form $\cos x + \sqrt{-1}\sin x$, and therefore of all imaginary expressions, is reduced to the addition of arcs under the circular functions, and the involution of such quantities to the multiplication of the arcs.

59.] Equivalent expressions of $(\sin x)^n$ and $(\cos x)^n$, in terms of the sines and cosines of the multiple arcs.

To abbreviate the notation, let us substitute as follows:

$$\left.\begin{array}{l} e^{x\sqrt{-1}} = z, \\ e^{-x\sqrt{-1}} = \dfrac{1}{z}, \end{array}\right\} \quad \therefore \quad \left.\begin{array}{l} e^{mx\sqrt{-1}} = z^m, \\ e^{-mx\sqrt{-1}} = \dfrac{1}{z^m}; \end{array}\right\} \qquad (41)$$

$$\therefore \quad 2\cos x = z + \frac{1}{z}, \qquad 2\cos mx = z^m + \frac{1}{z^m}, \qquad (42)$$

$$2\sqrt{-1}\sin x = z - \frac{1}{z}, \quad 2\sqrt{-1}\sin mx = z^m - \frac{1}{z^m}. \qquad (43)$$

Since therefore $\quad 2\cos x = z + \dfrac{1}{z},$

$$\therefore \quad 2^n (\cos x)^n = \left(z + \frac{1}{z}\right)^n$$

$$= z^n + nz^{n-2} + \frac{n(n-1)}{1.2} z^{n-4} + \ldots\ldots$$

$$\ldots\ldots + \frac{n(n-1)}{1.2} \frac{1}{z^{n-4}} + n\frac{1}{z^{n-2}} + \frac{1}{z^n},$$

$$= z^n + \frac{1}{z^n} + n\left(z^{n-2} + \frac{1}{z^{n-2}}\right)$$

$$+ \frac{n(n-1)}{1.2}\left(z^{n-4} + \frac{1}{z^{n-4}}\right) + \ldots\ldots \qquad (44)$$

$$= 2\cos nx + 2n\cos(n-2)x + 2\frac{n(n-1)}{1.2}\cos(n-4)x + \ldots$$

$$\therefore \quad (\cos x)^n = \frac{1}{2^{n-1}}\left\{\cos nx + n\cos(n-2)x\right.$$

$$\left. + \frac{n(n-1)}{1.2}\cos(n-4)x + \ldots\ldots\right\} \qquad (45)$$

If n be even, say $= 2r$, there are $2r + 1$ terms in series (44), $2r$ of which will give rise to r different cosines, viz. $\cos nx$, $\cos(n-2)x$, ... $\cos 2x$; and the remaining term, which is the middle one of the series, is independent of x, viz.:

$$\frac{n(n-1)(n-2)\ldots\ldots\left(\frac{n}{2}+2\right)\left(\frac{n}{2}+1\right)}{1.2.3\ldots\ldots\quad\left(\frac{n}{2}-1\right)\ \frac{n}{2}}.$$

But if n be odd, say $= 2r + 1$, the series will have $2r + 2$ terms, which may be combined into pairs; from each of which a cosine will arise, and there will be $\dfrac{n+1}{2}$ different cosines, viz. $\cos nx$, $\cos(n-2)x$, ... $\cos 3x$, $\cos x$.

Two examples are subjoined to illustrate the method of expansion.

Ex. 1. To expand $(\cos x)^6$ in terms of cosines of multiple arcs.

$$\because \quad 2\cos x = z + \frac{1}{z},$$

$$2^6 (\cos x)^6 = z^6 + 6z^4 + 15z^2 + 20 + \frac{15}{z^2} + \frac{6}{z^4} + \frac{1}{z^6},$$

$$= \left(z^6 + \frac{1}{z^6}\right) + 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) + 20,$$

$$= 2\cos 6x + 12\cos 4x + 30\cos 2x + 20,$$

$$\therefore \quad (\cos x)^6 = \frac{1}{2^5}\left\{\cos 6x + 6\cos 4x + 15\cos 2x + 10\right\}.$$

Ex. 2. To expand $(\cos x)^5$ in terms of cosines of multiple arcs.

$$2\cos x = z + \frac{1}{z},$$

$$\therefore \quad 2^5 (\cos x)^5 = \left(z^5 + \frac{1}{z^5}\right) + 5\left(z^3 + \frac{1}{z^3}\right) + 10\left(z + \frac{1}{z}\right),$$

$$= 2\cos 5x + 10\cos 3x + 20\cos x,$$

$$\therefore \quad (\cos x)^5 = \frac{1}{2^4}\left\{\cos 5x + 5\cos 3x + 10\cos x\right\}.$$

Again, since

$$2\sqrt{-1}\,\sin x = z - \frac{1}{z},$$

$$2^n(-1)^{\frac{n}{2}} (\sin x)^n = \left(z - \frac{1}{z}\right)^n,$$

$$= z^n - nz^{n-2} + \frac{n(n-1)}{1.2} z^{n-4} - \ldots\ldots$$

$$(-)^{n-2}\frac{n(n-1)}{1.2}\frac{1}{z^{n-4}}\ (-)^{n-1} n \frac{1}{z^{n-2}}\ (-)^n \frac{1}{z^n},$$

$$= \left(z^n\,(-)^n \frac{1}{z^n}\right) - n\left(z^{n-2}\,(-)^{n-2}\frac{1}{z^{n-2}}\right)$$

$$+ \frac{n(n-1)}{1.2}\left(z^{n-4}\,(-)^{n-4}\frac{1}{z^{n-4}}\right) - \ldots\ldots \qquad (46)$$

of which series there will be four cases according as n is of one or other of the forms $4r$, $4r+1$, $4r+2$, $4r+3$; but it is not worth while to write down the general terms of the series, as particular examples had better be solved independently. But it is to be observed that, if n be even, the series (46) involves

cosines only; whereas, if n be odd, it involves sines only. Two examples are subjoined.

Ex. 1. To expand $(\sin x)^5$ in terms of sines of the multiple arcs.

$$2\sqrt{-1}\,\sin x = z - \frac{1}{z},$$

$$\begin{aligned}2^5\sqrt{-1}\,(\sin x)^5 &= z^5 - 5z^3 + 10z - 10\frac{1}{z} + 5\frac{1}{z^3} - \frac{1}{z^5},\\ &= \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z + \frac{1}{z}\right),\\ &= 2\sqrt{-1}\sin 5x - 10\sqrt{-1}\sin 3x + 20\sqrt{-1}\sin x,\end{aligned}$$

$$\therefore\quad (\sin x)^5 = \frac{1}{2^4}\left\{\sin 5x - 5\sin 3x + 10\sin x\right\}.$$

Ex. 2. To expand $(\sin x)^6$ in terms of cosines of the multiple arcs.

$$2\sqrt{-1}\,\sin x = z - \frac{1}{z},$$

$$\begin{aligned}\therefore\quad -2^6(\sin x)^6 &= \left(z^6 + \frac{1}{z^6}\right) - 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) - 20,\\ &= 2\cos 6x - 12\cos 4x + 30\cos 2x - 20,\end{aligned}$$

$$\therefore\quad (\sin x)^6 = \frac{1}{2^5}\left\{-\cos 6x + 6\cos 4x - 15\cos 2x + 10\right\}.$$

60.] The Resolution of $x^n - 1 = 0$, and $x^n + 1 = 0$ into their factors.

Since by equation (40), Art. 58,

$$(\cos x \pm \sqrt{-1}\,\sin x)^m = \cos mx \pm \sqrt{-1}\,\sin mx, \tag{47}$$

and since this equation is true, whether m be positive or negative, integral or fractional, for m let $\frac{1}{n}$ be substituted, and for the general symbol x, let $x + 2k\pi$ be written, *k being a whole number*, so that (47) becomes

$$\begin{aligned}&\{\cos(x + 2k\pi) \pm \sqrt{-1}\,\sin(x + 2k\pi)\}^{\frac{1}{n}}\\ &\qquad = \cos\frac{x + 2k\pi}{n} \pm \sqrt{-1}\,\sin\frac{x + 2k\pi}{n}.\end{aligned} \tag{48}$$

Firstly in this equation let $x = 0$; therefore, as $\cos 2k\pi = 1$ and $\sin 2k\pi = 0$, we have

$$(1)^{\frac{1}{n}} = \cos\frac{2k\pi}{n} \pm \sqrt{-1}\sin\frac{2k\pi}{n} \tag{49}$$

but if $x^n - 1 = 0$, $x = 1^{\frac{1}{n}}$, and therefore the several values of x, or roots of the equation $x^n - 1 = 0$, are the values which the right-hand member of equation (49) admits of. Now k may be any whole number; let therefore

$$\left.\begin{array}{llll} k = 0 & \ldots \therefore & 1^{\frac{1}{n}} = \cos 0 \pm \sqrt{-1}\sin 0 = 1, \\ k = 1 & \ldots & 1^{\frac{1}{n}} = \cos\frac{2\pi}{n} \pm \sqrt{-1}\sin\frac{2\pi}{n}, \\ k = 2 & \ldots & 1^{\frac{1}{n}} = \cos\frac{4\pi}{n} \pm \sqrt{-1}\sin\frac{4\pi}{n}, \\ \ldots & \ldots & \ldots \\ \multicolumn{3}{l}{\text{and so on, until, if } n \text{ be even,}} \\ k = \frac{n}{2} - 1 & \ldots & 1^{\frac{1}{n}} = \cos\frac{n-2}{n}\pi \pm \sqrt{-1}\sin\frac{n-2}{n}\pi, \\ k = \frac{n}{2} & \ldots & 1^{\frac{1}{n}} = \cos\pi \pm \sqrt{-1}\sin\pi = -1. \end{array}\right\} \tag{50}$$

After which term the values recur, for the substitution of $\frac{n}{2} + 1$ for k gives the same values of $1^{\frac{1}{n}}$ as the substitution of $\frac{n}{2} - 1$.

And if n be odd, the substitutions for k must continue until

$$k = \frac{n-1}{2} \quad \therefore \; (1)^{\frac{1}{n}} = \cos\frac{n-1}{n}\pi \pm \sqrt{-1}\sin\frac{n-1}{n}\pi, \tag{51}$$

after which the values recur; and thus in both cases we have n, and only n, different values of $1^{\frac{1}{n}}$: and therefore $x^n - 1 = 0$ has n, and only n, roots.

And thus we may resolve $x^n - 1$ into its factors; for observing the roots in group (50), corresponding to $k = 0$, there is a factor $x - 1$; corresponding to $k = 1$, there are two factors, viz.

$$x - \left(\cos\frac{2\pi}{n} + \sqrt{-1}\sin\frac{2\pi}{n}\right), \text{ and } x - \left(\cos\frac{2\pi}{n} - \sqrt{-1}\sin\frac{2\pi}{n}\right),$$

which may be compounded into a quadratic factor

$$x^2 - 2x\cos\frac{2\pi}{n} + 1.$$

Similarly may each of the other pairs of simple factors be compounded into a quadratic factor; and the last factor, when n is even, is $x + 1$. So that

$$x^n - 1 = (x-1)\left(x^2 - 2x\cos\frac{2\pi}{n} + 1\right)\left(x^2 - 2x\cos\frac{4\pi}{n} + 1\right)$$
$$\ldots\ldots\left(x^2 - 2x\cos\frac{n-2}{n}\pi + 1\right)(x+1). \tag{52}$$

Similarly when n is odd:

$$x^n - 1 = (x-1)\left(x^2 - 2x\cos\frac{2\pi}{n} + 1\right)\left(x^2 - 2x\cos\frac{4\pi}{n} + 1\right)$$
$$\ldots\ldots\left(x^2 - 2x\cos\frac{n-1}{n}\pi + 1\right). \tag{53}$$

Secondly, in equation (48), let $x = \pi$; then, since $\cos(2k+1)\pi = -1$, and $\sin(2k+1)\pi = 0$,

$$(-1)^{\frac{1}{n}} = \cos\frac{2k+1}{n}\pi \pm \sqrt{-1}\sin\frac{2k+1}{n}\pi. \tag{54}$$

But if $x^n + 1 = 0$, $x = (-1)^{\frac{1}{n}}$, and therefore the several values of x, or roots of the equation $x^n + 1 = 0$, are the values which the right-hand member of equation (54) admits of. Now k may be any integral number; let therefore

$$\left.\begin{array}{lll} k = 0 & \therefore\ (-1)^{\frac{1}{n}} = \cos\frac{\pi}{n} \pm \sqrt{-1}\sin\frac{\pi}{n}, \\ k = 1 & (-1)^{\frac{1}{n}} = \cos\frac{3\pi}{n} \pm \sqrt{-1}\sin\frac{3\pi}{n}, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \text{until, when } n \text{ is even,} & \\ k = \frac{n}{2} - 1 & (-1)^{\frac{1}{n}} = \cos\frac{n-1}{n}\pi \pm \sqrt{-1}\sin\frac{n-1}{n}\pi; \end{array}\right\} \tag{55}$$

after which the values recur, for the substitution of $\frac{n}{2}$ for k gives the same values of $(-1)^{\frac{1}{n}}$ as the substitution of $\frac{n}{2} - 1$;

and the substitution of $\frac{n}{2}+1$ for k, gives the same values of $(-1)^{\frac{1}{n}}$ as the substitution of $\frac{n}{2}-2$.

And if n be odd, the substitutions for k must continue until

$$\left.\begin{aligned} k &= \frac{n-3}{2} \quad \therefore\ (-1)^{\frac{1}{n}} = \cos\frac{n-2}{n}\pi \pm \sqrt{-1}\sin\frac{n-2}{n}\pi, \\ k &= \frac{n-1}{2} \quad\ \ (-1)^{\frac{1}{n}} = \cos\pi \pm \sqrt{-1}\sin\pi = -1, \end{aligned}\right\} \quad (56)$$

after which the values recur; and thus in both cases we have n, and only n, different values of $(-1)^{\frac{1}{n}}$; and therefore $x^n+1=0$ has n, and only n, roots.

We may thus resolve $x^n+1=0$ into its factors; for observing the roots in group (55), corresponding to $k=0$, there are two factors, viz. $x-\left(\cos\frac{\pi}{n}+\sqrt{-1}\sin\frac{\pi}{n}\right)$, and $x-\left(\cos\frac{\pi}{n}-\sqrt{-1}\sin\frac{\pi}{n}\right)$, which may be compounded into a quadratic factor $x^2-2x\cos\frac{\pi}{n}+1$. Similarly may each of the other pairs of factors corresponding to $k=1$, $k=2$, ... $k=\frac{n}{2}-1$ be compounded into a quadratic factor; so that, when n is even,

$$x^n+1 = \left(x^2-2x\cos\frac{\pi}{n}+1\right)\left(x^2-2x\cos\frac{3\pi}{n}+1\right)\ \ldots\ldots$$
$$\ldots\ldots\ \left(x^2-2x\cos\frac{n-1}{n}\pi+1\right). \quad (57)$$

Similarly, when n is odd, there are quadratic factors corresponding to $k=0$, $k=1$, $k=2$, ... $k=\frac{n-3}{2}$; and when $k=\frac{n-1}{2}$, as appears by (56), there is a simple factor $x+1$, so that

$$x^n+1 = \left(x^2-2x\cos\frac{\pi}{n}+1\right)\left(x^2-2x\cos\frac{3\pi}{n}+1\right)\ \ldots\ldots$$
$$\ldots\ldots\ \left(x^2-2x\cos\frac{n-2}{2}\pi+1\right)(x+1). \quad (58)$$

61.] If the problem be to resolve $x^n - a^n$, and $x^n + a^n$ into their factors, then, since

$$x^n - a^n = 0, \qquad \text{and} \qquad x^n + a^n = 0,$$

$$x = a\,(1)^{\frac{1}{n}}, \qquad \text{and} \qquad x = a\,(-1)^{\frac{1}{n}};$$

whence, in both cases, there will be n different values of x, arising from the n different values of $(1)^{\frac{1}{n}}$ and of $(-1)^{\frac{1}{n}}$, which have been found above.

There are very many useful and curious properties of the roots of $+1$ and of -1, which are explained in treatises on the Theory of Equations; but which it would be beside our object to discuss, as the above resolution is all that we shall require in the subsequent parts of our work.

62.] On the impossible Logarithms of positive and negative numbers.

One of the most remarkable results of our admission into the subjects of the Calculus of such quantities and symbols as we are now considering, is the extension that they afford to the theory of logarithms.

Since, by equation (31), Art. 58,

$$\cos x + \sqrt{-1}\,\sin x = e^{x\sqrt{-1}},$$

for x let us write $x + 2k\pi$, k being any integer,

$$\therefore\ \cos(x+2k\pi) + \sqrt{-1}\,\sin(x+2k\pi) = e^{(x+2k\pi)\sqrt{-1}}. \qquad (59)$$

$$\text{Let} \qquad x = 0 \qquad \therefore\ 1 = e^{2k\pi\sqrt{-1}}, \qquad (60)$$

$$x = \pi \qquad -1 = e^{(2k+1)\pi\sqrt{-1}}; \qquad (61)$$

in which remarkable results it must be remembered that e and π are severally the symbols of the arithmetical numbers 2.7182818 and 3.14159. Therefore from (60) it follows, that $2k\pi\sqrt{-1}$ is the *general* Napierian logarithm of 1, and $(2k+1)\pi\sqrt{-1}$ of -1; and therefore

$$2k\pi\sqrt{-1} = \log_e 1, \qquad (62)$$

$$(2k+1)\pi\sqrt{-1} = \log_e(-1). \qquad (63)$$

In (62), if $k = 0$, $\log_e 1 = 0$, which is the common *arithmetical* logarithm of 1; but as k may be any integral number,

it follows that $+1$ has an infinite number of Napierian logarithms, all of which, except 0, are affected with $\sqrt{-1}$; hence also it follows, that every positive number has an infinite number of logarithms to the same base. For suppose x to be the arithmetical Napierian logarithm of y, so that

$$y = e^x,$$

$$\therefore \quad y = e^x \times 1 = e^x \times e^{2k\pi\sqrt{-1}} = e^{(x+2k\pi\sqrt{-1})}; \tag{64}$$

therefore $x+2k\pi\sqrt{-1}$ is the general Napierian logarithm of y, which we will represent by $\text{Log}_e y$, and represent the arithmetical logarithm by $\log_e y$;

$$\therefore \quad \text{Log}_e y = \log_e y + 2k\pi\sqrt{-1}. \tag{65}$$

Hence the arithmetical logarithm is the particular value of the general logarithm corresponding to $k = 0$.

Also since $\text{Log}_a y = \dfrac{\text{Log}_e y}{\text{Log}_e a}$, $\text{Log}_a y = \log_a y + \dfrac{2k\pi\sqrt{-1}}{\text{Log}_e a}$; (66)

and therefore, whatever be the base, every number has an infinite number of logarithms to that base.

Again, in equation (63), since k is to be an integer, it follows, that $\log_e(-1)$ can never be a possible quantity, and therefore -1 has no arithmetical logarithm; yet, since k may be any integer, every negative number has an infinite number of Napierian logarithms, and therefore of logarithms to any other base, all of which are affected with $\sqrt{-1}$.

In equation (63), let $k = 0$,

$$\therefore \quad \pi\sqrt{-1} = \log_e(-1); \tag{67}$$

and more generally,

$$\pi = \frac{\log_e(-1)}{(2k+1)\sqrt{-1}}; \tag{68}$$

two of the most curious results in Analysis; in which however, and in all similar expressions, we must bear in mind that e and π are the symbolical representations of certain series, and are therefore to be interpreted with respect to them; and in an algebraical system of course, which admits them amongst those quantities whose laws it takes cognizance of.

Section 3.—*Theory of the Equicrescent Variable, and Taylor's Series.*

63.] In the two preceding Sections of this Chapter we have considered certain theorems which arise from an *algebraical* relation between $f(x)$ and its several and successive derived-functions; but as these have been formed on an unexplained hypothesis of x increasing by equal increments, by virtue of which x has been called the *equicrescent* variable, an adequate conception of the subject requires a discussion at greater length. We shall reserve other important *algebraical* relations to the next Chapter.

Let $y = f(x)$ be an explicit function of x, finite and continuous for all values of x under which we consider it; whether any other values may make it infinite or discontinuous is what we are not concerned with.

Let x receive an increment dx, so that, in accordance with Art. 17, y becomes $y + dy$, and we have

$$y + dy = f(x + dx). \tag{69}$$

And let x receive another increment, not necessarily equal to dx, and let it be $dx + d.dx = dx + d^2x$; so that we have

$$\begin{aligned} y + dy + d(y + dy) &= f\{x + dx + d(x + dx)\}, \\ y + 2dy + d^2y &= f(x + 2dx + d^2x). \end{aligned} \tag{70}$$

And again, let x receive an increment not equal to the former ones; and so on, whereby a series of equations will be formed:

$$\left.\begin{aligned} y &= f(x), \\ y + dy &= f(x + dx), \\ y + 2dy + d^2y &= f(x + 2dx + d^2x), \\ y + 3dy + 3d^2y + d^3y &= f(x + 3dx + 3d^2x + d^3x), \\ \cdots\cdots & \cdots\cdots\cdots\cdots \\ y + ndy + \frac{n(n-1)}{1.2}d^2y + \frac{n(n-1)(n-2)}{1.2.3}d^3y &+ \ldots\ldots \\ = f\Big\{x + ndx + \frac{n(n-1)}{1.2}d^2x &+ \ldots\ldots\Big\}, \end{aligned}\right\} \tag{71}$$

the last general expression in which group may be shewn to be true for all positive and integral values of n, by a method similar to that employed in Art. 53.

Hence are determined the several and successive complete variations of an explicit function of x, corresponding to the successive values of the variable; but the general result is much shortened by making x equicrescent*.

64.] Let all the dxs be equal, so that d^2x, which is the increment of one dx over another, is equal to zero; and similarly

$$d^3x = d^4x = \quad . \quad . \quad . = d^n x = \quad . \quad . \quad . = 0;$$

whereby the equation becomes

$$f(x+ndx) = y+ndy+\frac{n(n-1)}{1.2}d^2y+\frac{n(n-1)(n-2)}{1.2.3}d^3y+..(72)$$

Thus the distinguishing character of an equicrescent variable is that all its differentials after the first vanish. And as this condition is of the greatest importance in the application of the Calculus to questions of Geometry and Physics, it is advisable to illustrate it before we proceed to discuss its other properties.

Suppose that we are considering any function of x between the values x_n and x_0, x_n being the greater of the two, and the function remaining finite and continuous for all values of x between these limits; let us resolve the difference $x_n - x_0$ into small elements, the number of them being of course infinitely large when each element is infinitesimal; let dx be the type of each. It is at once manifest that all the elements need not be equal; that is, all the dxs are not necessarily equal. And if they be not, there will be an increment of one over another; that is, there will be a d^2x. Neither again need all the d^2xs be equal; but if they be not, there will be an increment of one d^2x over another d^2x; that is, there will be a $d.d^2x$ or a d^3x.

* The variable, which I have ventured to call Equicrescent, and thus to coin a new word for, is by most writers called "Independent," and by some old ones "Principal Variable;" to the latter term the objection is, that it does not express the characteristic property of the thing to which it is applied, and has in fact no pretension to appropriateness of nomenclature; the former term is by all writers, and in the present treatise, used in a different signification, viz.: to express that variable which first changes value, and due to the change of which the other variables change, and are therefore called *dependent*: see Art. 12; and as such an independent variable may or may not be equicrescent, it is inconvenient to use the same term in two different senses: and especially as the term does not express that character of the variable which renders a distinctive appellation desirable.

And similarly with regard to the other differentials. But if we once introduce the condition that a differential of any order shall be resolved into elements which are all equal to one another, then all the subsequent differentials vanish; and thus, if all the dxs are equal, as above,

$$d^2x = d^3x = \ldots\ldots = 0.$$

Or again, conceive a small body, as a billiard-ball, to move over a finite distance in a straight line in a finite time; consider the straight line to be the axis of x; let the body at the beginning of the motion be at a distance x_0 from the origin, and at the end of the time be at a distance x_n, and conceive the time of its passing over the distance $x_n - x_0$ to be t; resolve this time into equal elements dt, and the space $x_n - x_0$ into corresponding elements, of each of which the type is dx. If the body moves through the whole space at the same rate, viz. with the same velocity, then, during equal times dt, equal spaces dx will be described; but, if the velocity varies, equal spaces will not be passed over in equal times. On the first supposition all the dxs will be equal, therefore $d^2x = 0$, and x is an equicrescent variable; on the second the dxs vary, and d^2x, which is the increment of one dx passed over in a time dt, over another dx passed over in the preceding or succeeding time dt, as the case may be, is the measure of the increase of the rate of motion. If then all the d^2xs were equal, we should say that the velocity of motion is continually increasing, and at a constant rate; but if d^2x were not constant, then the rate of increase of the velocity of the ball is no longer constant, but varies according to some law on which the rate of increase depends. It will be observed, however, that if the whole time of motion be resolved into equal elements dt, the supposition of x being equicrescent is incompatible with a varying velocity. Hence too it is manifest, that generally we are not at liberty to consider more than one of the variables to increase or decrease by equal augments; as in the case above, if we resolve the time into equal elements, then, in general, unequal spaces will be passed over in equal times, and we cannot consider all the dxs to be equal, and therefore we cannot make $d^2x = 0$; and if we resolve the distance into equal parts, then, if the velocity varies, these equal spaces will be passed over in unequal times, and therefore all the dts will not be equal, and

we cannot put $d^2t = 0$. In general, however, we are at liberty to choose for an equicrescent variable whichever is most convenient.

65.] Let us now consider in what manner such considerations as these modify the equations of derived-functions in Art. 52. In the series there given we have

$$y = f(x),$$
$$dy = f'(x)\, dx.$$

Now considering $f'(x)\, dx$ to be the product of two variable quantities, and differentiating it as such, and, in accordance with the former notation, making $f''(x)\, dx$ to be the symbol for $d.f'(x)$ and $f'''(x)\, dx$ for $d.f''(x)$, and so on, we have

$$\begin{aligned} d^2y &= f''(x)\, dx^2 + f'(x)\, d^2x, \\ d^3y &= f'''(x)\, dx^3 + 3f''(x)\, dx\, d^2x + f'(x)\, d^3x, \\ d^4y &= f''''(x)\, dx^4 + 6f'''(x)\, dx^2\, d^2x + 3f''(x)\, (d^2x)^2 \\ &\quad + 4f''(x)\, dx\, d^3x + f'(x)\, d^4x, \end{aligned}$$

. . . .

Now let dx be constant; whence $d^2x = 0$, $d^3x = 0, \ldots\ldots$

$$\begin{aligned} \therefore\quad dy &= f'(x)\, dx, \\ d^2y &= f''(x)\, dx^2, \\ d^3y &= f'''(x)\, dx^3, \\ d^4x &= f''''(x)\, dx^4; \end{aligned}$$

$\therefore$ $f''(x)$, or its equivalent $\dfrac{d^2y}{dx^2}$, is derived from $f'(x)$, on the supposition that x is the equicrescent variable;

$f'''(x)$, or its equivalent $\dfrac{d^3y}{dx^3}$, is derived from $f''(x)$, on the same supposition;

and $f^n(x) = \dfrac{d^ny}{dx^n}$ is derived from $f^{n-1}(x)$, on the same supposition.

Whenever therefore we meet with these or similar symbols, it is to be borne in mind that they have been successively derived on the supposition that the variable x, that is the variable in the denominator, is equicrescent.

66.] To return to equation (72), Art. 64. Let $n\,dx = h$, h being a finite quantity, so that dx being an infinitesimal, n is an infinity of the same order; whence we have

$$n = \frac{h}{dx}; \tag{73}$$

and by substitution in equation (72),

$$f(x+h) = y + \frac{h}{dx}\,dy + \frac{\frac{h}{dx}\left(\frac{h}{dx}-1\right)}{1.2}\,d^2y$$

$$+ \frac{\frac{h}{dx}\left(\frac{h}{dx}-1\right)\left(\frac{h}{dx}-2\right)}{1.2.3}\,d^3y + \ldots\ldots \tag{74}$$

$$f(x+h) = y + h\frac{dy}{dx} + \frac{h\,(h-dx)}{1.2}\,\frac{d^2y}{dx^2}$$

$$+ \frac{h(h-dx)\,(h-2dx)}{1.2.3}\,\frac{d^3y}{dx^3} + \ldots\ldots \tag{75}$$

and as dx is infinitesimal, we must omit in the several numerators the infinitesimals which are subtracted from the finite quantities, and we may replace y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ ...... by $f(x)$, $f'(x)$, $f''(x)$, ... ; whence

$$f(x+h) = f(x) + h\,f'(x) + \frac{h^2}{1.2}f''(x) + \frac{h^3}{1.2.3}f'''(x) + \ldots$$

$$\ldots\ldots + \frac{h^n}{1.2.3\ldots n}f^n(x) + \ldots\ldots \tag{76}$$

A series known by the name of Taylor's Series, having been discovered by Dr. Brook Taylor, and first given in his "Methodus Incrementorum" in the year 1715; but as it is of the utmost importance that the conditions and extent of its applicability should be accurately determined, another and more exact proof will be given hereafter; and the above must be considered in the light of a presumption, that such a relation as equation (76) is likely to be true.

67.] In certain cases wherein the derived-functions ultimately vanish, the number of terms of the series (76) is finite; but generally the successively-derived functions are functions of x, and the series is continued to an infinite number of terms;

but the sum of all the terms after the nth may be expressed as follows by an algebraical formula. Since

$$f(x+h) = f(x) + \frac{h}{1}f'(x) + \frac{h^2}{1.2}f''(x) + \dots\dots$$
$$\dots\dots + \frac{h^{n-1}}{1.2.3\dots(n-1)}f^{n-1}(x) + \frac{h^n}{1.2.3\dots n}f^n(x)$$
$$+ \frac{h^{n+1}}{1.2.3\dots(n+1)}f^{n+1}(x) + \dots\dots \quad (77)$$

Therefore the sum of all the terms after the nth is equal to

$$\frac{h^n}{1.2.3\dots n}f^n(x) + \frac{h^{n+1}}{1.2.3\dots(n+1)}f^{n+1}(x) + \dots\dots$$
$$= \frac{h^n}{1.2.3\dots n}\left\{f^n(x) + \frac{h}{n+1}f^{n+1}(x) + \frac{h^2}{(n+1)(n+2)}f^{n+2}(x) + \right\} (78)$$
$$= \frac{h^n}{1.2.3\dots n}\left\{\text{some quantity} > f^n(x) \text{ and} < f^n(x+h)\right\} \qquad (79)$$

the latter factor of (78) is I say greater than $f^n(x)$, because the algebraical sum of such a series of terms is greater than its first term; and it is less than $f^n(x+h)$, because if equation (77) be derivated n times, there results

$$f^n(x+h) = f^n(x) + \frac{h}{1}f^{n+1}(x) + \frac{h^2}{1.2}f^{n+2}(x) + \frac{h^3}{1.2.3}f^{n+3}(x) +$$

and with the exception of the first term of this series, every term is greater than the corresponding term of the series in the latter factor of (78), because, the numerator of the fractions being the same, the denominators are severally less. Hence representing by θ, *some* positive and proper fraction, we may symbolize (79) by

$$\frac{h^n}{1.2.3\dots n}f^n(x+\theta h) \qquad (80)$$

for (80) is too small when $\theta = 0$, and is too large when $\theta = 1$; hence *some* value of θ between 0 and 1 will give the correct value. Hence Taylor's Series becomes

$$f(x+h) = f(x) + \frac{h}{1}f'(x) + \frac{h^2}{1.2}f''(x) + \dots\dots$$
$$\dots\dots + \frac{h^n}{1.2.3\dots n}f^n(x+\theta h) \quad (81)$$

and as the right-hand side of the equation has a determinate number of terms, the only difference in the absolute equality of the two sides of the equation is that which arises from θ being an undetermined fraction, *mean* between 0 and 1.

68.] Examples of Taylor's Series.

Ex. 1.
$$
\begin{aligned}
f(x) &= \log_e x, \\
f'(x) &= \frac{1}{x} = x^{-1}, \\
f''(x) &= (-)\, x^{-2}, \\
f'''(x) &= (-)^2\, 1.2\, x^{-3}, \\
&\quad \cdots\cdots\cdots \\
f^n(x) &= (-)^{n-1}\, 1.2.3 \ldots\ldots (n-1)\, x^{-n};
\end{aligned}
$$

$$
\log_e(x+h) = \log_e x + \frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \ldots\ldots
$$
$$
(-)^{n-2}\frac{h^{n-1}}{(n-1)\,x^{n-1}}\,(-)^{n-1}\,\frac{h^n}{n}\,\frac{1}{(x+\theta h)^n} \qquad (82)
$$

Ex. 2. $\quad f(x) = \sin x;$

taking the several derived-functions as calculated in Ex. 4, Art. 52, we have

$$
\sin(x+h) = \sin x + \cos x\,\frac{h}{1} - \sin x\,\frac{h^2}{1.2} + \ldots\ldots
$$
$$
\ldots\ldots + \sin\left\{x+(n-1)\frac{\pi}{2}\right\}\frac{h^{n-1}}{1.2.3\ldots(n-1)}
$$
$$
+ \frac{h^n}{1.2.3\ldots n}\sin\left(x+\theta h + n\frac{\pi}{2}\right) \qquad (83)
$$

Ex. 3. $\quad f(x) = \tan^{-1} x;$

$$
\therefore\ \tan^{-1}(x+h) = \tan^{-1} x + \frac{1}{1+x^2}\,\frac{h}{1} - \frac{2x}{(1+x^2)^2}\,\frac{h^2}{1.2}
$$
$$
+ \frac{2(3x^2-1)}{(1+x^2)^3}\,\frac{h^3}{1.2.3} + \ldots\ldots \qquad (84)
$$

The general term is too complicated to be of any practical use, though it may without difficulty be symbolically expressed by the method of Ex. 5, Art. 52.

Ex. 4. Given $f(x+y) = f(x)+f(y)$, where x and y are independent of each other, it is required to find the form of the function.

Expanding $f(x+y)$ by Taylor's Series, we have

$$f(x)+f(y) = f(x+y),$$

$$= f(x) + f'(x)\frac{y}{1} + f''(x)\frac{y^2}{1.2} + f'''(x)\frac{y^3}{1.2.3} + \dots\dots$$

$$\therefore \; f(y) = f'(x)\frac{y}{1} + f''(x)\frac{y^2}{1.2} + f'''(x)\frac{y^3}{1.2.3} + \dots\dots$$

As x and y are independent of each other, the form of the function of y does not depend on x; therefore x and all functions of x are constant with respect to y. Hence we may put

$$f'(x) = \text{a constant} = a;$$

$$\therefore \; f''(x) = f'''(x) = \dots\dots = 0;$$

$$\therefore \; f(y) = ay; \text{ whence also } f(x) = ax,$$

$$\text{and } f(x+y) = a(x+y);$$

a result which palpably satisfies the required conditions.

Ex. 5. $f(x+y) = f(x)\times f(y)$, where x and y are independent, to determine the form of the function.

Expanding as before, we have

$$f(x)\times f(y) = f(x) + f'(x)\frac{y}{1} + f''(x)\frac{y^2}{1.2} + f'''(x)\frac{y^3}{1.2.3} + \dots$$

$$f(y) = 1 + \frac{f'(x)}{f(x)}\frac{y}{1} + \frac{f''(x)}{f(x)}\frac{y^2}{1.2} + \frac{f'''(x)}{f(x)}\frac{y^3}{1.2.3} + \dots\dots$$

For the same reason as in the last example, let

$$\frac{f'(x)}{f(x)} = a; \qquad \therefore \; f'(x) = af(x),$$

$$\therefore \; f''(x) = af'(x) = a^2 f(x), \qquad \therefore \; \frac{f''(x)}{f(x)} = a^2.$$

Similarly $\dfrac{f'''(x)}{f(x)} = a^3$, &c.

$$\therefore \; f(y) = 1 + \frac{ay}{1} + \frac{a^2y^2}{1.2} + \frac{a^3y^3}{1.2.3} + \dots\dots$$

$$= e^{ay};$$

$$\therefore \quad \text{also } f(x) = e^{ax} \quad \text{and} \quad f(x+y) = e^{a(x+y)},$$

$$\therefore \quad f(x+y) = e^{ax} \times e^{ay} = f(x) \times f(y),$$

which satisfies the required condition.

By a similar process let the student prove that, if

$$f(x+y) + f(x-y) = 2f(x) \times f(y), \qquad f(x) = \cos ax;$$

$$\text{and if} \quad f(xy) = f(x) + f(y), \qquad f(x) = \log_a x.$$

SECTION 4.—*Change of the Equicrescent Variable.*

69.] From the supposition then which we are at liberty to make of one of the variables involved in an equation increasing by equal increments, and therefore of the several differentials of it, after the first, vanishing, problems such as the following arise.

(1.) Suppose $y = f(x)$, and that an expression is given involving x, y, and some of the derived-functions or differential coefficients which have been calculated on the supposition that one of the variables is equicrescent; to change the equation into its equivalent, when neither of the variables is equicrescent. Or

(2.) To transform it into its equivalent, when the other variable is equicrescent. Or

(3.) An expression being given involving a variable, which is either equicrescent or not, and its differentials, and also an equation being given connecting this variable with some other new variable; to eliminate the old variable and its differentials by means of these two equations, and to replace them in the original equation by their equivalents in terms of the new variable: the new variable being equicrescent or not, as the case may be. Or

(4.) It may be required to replace the variables and their differentials in a given differential expression, by their equivalents in terms of new variables, which are connected with the old variables by means of a sufficient number of given equations: the old and the new variables being equicrescent or not, as the case may be.

These several processes are called Changes of the equicrescent variable, and the method of effecting them is the same in all cases, viz.:

To replace the expression, which has been simplified by the condition of a variable being equicrescent, by its complete value when no supposition has been made, and then to introduce whatever other conditions the problem requires.

70.] Thus to solve the first two of the four cases above.

Let the given expression involve x, y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, ..., the differential coefficients (as their form indicates) having been calculated on the supposition that x is equicrescent; it is required to replace these several differential coefficients by their equivalents, when x is not equicrescent.

Since by equations (2) and (3), Art. 52,

$$\frac{d^2y}{dx^2} = \frac{d.f'(x)}{dx} \quad \text{and} \quad f'(x) = \frac{dy}{dx};$$

$\therefore$ if x be not equicrescent,

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{d^2y\,dx - d^2x\,dy}{dx^3}; \tag{85}$$

$$\frac{d^3y}{dx^3} = \frac{d\left(\frac{d^2y}{dx^2}\right)}{dx} = \frac{(d^3y\,dx - d^3x\,dy)dx - 3(d^2y\,dx - d^2x\,dy)d^2x}{dx^5} \tag{86}$$

and similarly the equivalents of the other differential coefficients may be determined; the second members of the equations (85) and (86) being the complete values when neither x nor y is equicrescent.

If x be equicrescent, the equations are identical.

If y be equicrescent, then $d^2y = d^3y = \ldots\ldots = 0$, and the original quantities must be replaced by their equivalents, viz.:

$$\frac{d^2y}{dx^2} \text{ must be replaced by } \ldots\ldots\ldots \quad -\frac{d^2x\,dy}{dx^3}, \tag{87}$$

$$\frac{d^3y}{dx^3} \ldots\ldots\ldots\ldots \quad \frac{3(d^2x)^2\,dy - d^3x\,dy\,dx}{dx^5}, \tag{88}$$

$$\frac{d^4y}{dx^4} \ldots\ldots \quad \frac{10d^3x\,d^2x\,dx - (dx)^2\,d^4x - 15(d^2x)^3}{dx^7}dy, \tag{89}$$

. . . .

The form however of the nth equivalent is too complicated to be of any use in the solution of a given example.

Ex. 1. To transform $x\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3-\frac{dy}{dx}=0$, into its equivalents, (α) when neither x nor y is equicrescent, (β) when y is equicrescent.

(α) Replace $\frac{d^2y}{dx^2}$ by its value as given in equation (85), and multiply by $(dx)^3$;

$$x(d^2y\,dx - d^2x\,dy) + dy^3 - dy\,dx^2 = 0.$$

(β) Let $d^2y = 0$, and we have

$$x\frac{d^2x}{dy^2} + \left(\frac{dx}{dy}\right)^2 - 1 = 0.$$

Ex. 2. To transform $(dy^2+dx^2)^{\frac{3}{2}}+adx\,d^2y = 0$, when x is equicrescent, into its equivalents, (α) when neither x nor y is equicrescent, (β) when y is equicrescent.

(α) Replace d^2y by $\frac{d^2y\,dx - d^2x\,dy}{dx}$, and we have

$$(dy^2 + dx^2)^{\frac{3}{2}} + a(d^2y\,dx - d^2x\,dy) = 0.$$

(β) And if y be equicrescent $d^2y = 0$, and we have

$$(dy^2 + dx^2)^{\frac{3}{2}} - a\,d^2x\,dy = 0,$$

$$\left\{1+\frac{dx^2}{dy^2}\right\}^{\frac{3}{2}} - a\frac{d^2x}{dy^2} = 0.$$

71.] In (3) and (4) of the cases of Art. 69; suppose the given expression to involve $y, x, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots\ldots$ wherein x is the equicrescent variable, and suppose the new equation to be of the form $x = f(\theta)$, and the problem to be the elimination of x and its differentials between these two equations, we must first replace $\frac{d^2y}{dx^2}\ldots\ldots$ by their complete values, and calculate dx, d^2x, $d^3x, \ldots\ldots$ in terms of $d\theta$, $d^2\theta$, $d^3\theta, \ldots\ldots$, and then we are at liberty to make any supposition that may be convenient as to y or θ being equicrescent. And a similar method (as is shewn in example 3. below) must be adopted when two equations are given connecting x and y with two new variables, say r and θ.

Ex. 1. **Eliminate x between**

$$\frac{d^2y}{dx^2} - \frac{x}{1-x^2}\frac{dy}{dx} + \frac{y}{1-x^2} = 0, \text{ and}$$

$$x = \cos\theta;$$

and simplify the result by making θ the equicrescent variable. Or the question may be put thus,

To transform $\frac{d^2y}{dx^2} - \frac{x}{1-x^2}\frac{dy}{dx} + \frac{y}{1-x^2} = 0$ into its equivalent, when $\theta(=\cos^{-1}x)$ is the equicrescent variable.

The above equation when complete is

$$\frac{d^2y\,dx - d^2x\,dy}{dx^3} - \frac{x}{1-x^2}\frac{dy}{dx} + \frac{y}{1-x^2} = 0,$$

$$x = \cos\theta; \qquad \therefore\quad dx = -\sin\theta\,d\theta,$$

$$d^2x = -\cos\theta\,(d\theta)^2 - \sin\theta\,d^2\theta.$$

But since θ is the equicrescent, $d^2\theta = 0$; whence, substituting, we have

$$\frac{-d^2y\sin\theta\,d\theta + \cos\theta\,(d\theta)^2\,dy}{-(\sin\theta)^3\,d\theta^3} + \frac{\cos\theta}{(\sin\theta)^2}\,\frac{dy}{\sin\theta\,d\theta} + \frac{y}{(\sin\theta)^2} = 0,$$

$$\frac{d^2y}{d\theta^2} + y = 0.$$

Ex. 2. Eliminate x between

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0, \text{ and}$$

$$x^2 = 4\theta,$$

and express the result; (α) when neither y nor θ is equicrescent, (β) when θ is equicrescent, (γ) when y is equicrescent.

The complete expression of the above equation is

$$\frac{d^2y\,dx - d^2x\,dy}{dx^3} + \frac{1}{x}\frac{dy}{dx} + y = 0,$$

$$x = 2(\theta)^{\frac{1}{2}}; \qquad \therefore\quad dx = \theta^{-\frac{1}{2}}\,d\theta,$$

$$d^2x = -\frac{(d\theta)^2}{2\theta^{\frac{3}{2}}} + \frac{d^2\theta}{\theta^{\frac{1}{2}}};$$

whence, by substitution,

(α) $y(d\theta)^3 + dy\,(d\theta)^2 + \theta\, d^2y\, d\theta - \theta\, dy\, d^2\theta = 0.$

(β) And if θ be equicrescent, $d^2\theta = 0$; whereby we have

$$y(d\theta)^3 + dy\,(d\theta)^2 + \theta\, d^2y\, d\theta = 0;$$

or

$$\theta\frac{d^2y}{d\theta^2} + \frac{dy}{d\theta} + y = 0.$$

(γ) And if y be equicrescent, $d^2y = 0$; whereby we have

$$\theta\frac{d^2\theta}{dy^2} - \left(\frac{d\theta}{dy}\right)^2 - y\left(\frac{d\theta}{dy}\right)^3 = 0.$$

Ex. 3. Eliminate x, y, dx and dy between

$$t = \frac{x\,dy - y\,dx}{y\,dy + x\,dx} \quad \text{and} \quad \begin{cases} x = r\cos\theta \\ y = r\sin\theta; \end{cases}$$

$$\therefore \quad dx = dr\cos\theta - r\sin\theta\, d\theta,$$
$$dy = dr\sin\theta + r\cos\theta\, d\theta;$$

$$\therefore \quad x\,dy - y\,dx = r^2 d\theta,$$
$$x\,dx + y\,dy = r\,dr;$$

$$\therefore \quad t = \frac{r\,d\theta}{dr}.$$

Ex. 4. Express in terms of r and θ, $\rho = \dfrac{\left\{1+\dfrac{dy^2}{dx^2}\right\}^{\frac{3}{2}}}{-\dfrac{d^2y}{dx^2}}$

having given $x = r\cos\theta$, and $y = r\sin\theta$, and state the result, (α) in the most general form, (β) when θ is equicrescent, (γ) when r is equicrescent.

The complete value of ρ is

$$\rho = \frac{(dx^2 + dy^2)^{\frac{3}{2}}}{d^2x\,dy - d^2y\,dx};$$

$$dx = dr\cos\theta - r\sin\theta\, d\theta,$$

$$dy = dr\sin\theta + r\cos\theta\, d\theta;$$

$$\therefore \quad d^2x = d^2r\cos\theta - 2\sin\theta\, dr\, d\theta - r\cos\theta\,(d\theta)^2 - r\sin\theta\, d^2\theta,$$

$$d^2y = d^2r\sin\theta + 2\cos\theta\, dr\, d\theta - r\sin\theta\,(d\theta)^2 + r\cos\theta\, d^2\theta;$$

$$\therefore \quad (dx)^2 + (dy)^2 = (dr)^2 + r^2(d\theta)^2,$$

$$d^2x\,dy - d^2y\,dx = r\,d^2r\,d\theta - 2(dr)^2 d\theta - r^2(d\theta)^3 - r\,dr\,d^2\theta.$$

$$(\alpha) \quad \therefore \quad \rho = \frac{\{(dr)^2 + r^2 (d\theta)^2\}^{\frac{3}{2}}}{r\, d^2r\, d\theta - 2(dr)^2 d\theta - r^3 (d\theta)^3 - r\, dr\, d^2\theta};$$

$$(\beta) \qquad = \frac{\left\{\left(\frac{dr}{d\theta}\right)^2 + r^2\right\}^{\frac{3}{2}}}{r\frac{d^2r}{d\theta^2} - 2\left(\frac{dr}{d\theta}\right)^2 - r^2};$$

$$(\gamma) \qquad = -\frac{\left\{1 + r^2\left(\frac{d\theta}{dr}\right)^2\right\}^{\frac{3}{2}}}{r\frac{d^2\theta}{dr^2} + r^2\left(\frac{d\theta}{dr}\right)^3 + 2\frac{d\theta}{dr}}.$$

SECTION 5.—*Successive Differentiation of Explicit Functions of two or more Independent Variables.*

72.] Let $u = \mathrm{F}(x, y, z, \ldots\ldots)$ be the type of the function of many variables which we are considering, and of which the successive differentials are to be found. But, to fix our thoughts, let us first take a function of two variables, viz.:

$$u = \mathrm{F}(x, y). \tag{90}$$

The first total differential of u is

$$\mathrm{D}u = \left(\frac{du}{dx}\right) dx + \left(\frac{du}{dy}\right) dy; \tag{91}$$

in which expression, $\left(\frac{du}{dx}\right)$ and $\left(\frac{du}{dy}\right)$, and therefore $\mathrm{D}u$, are in general functions of both x and y, and therefore admit of being differentiated again, as well in respect of all the variables, whereby successive total differentials are formed, as of any one of the variables, whence successive partial differentials arise. The total differential of $\mathrm{D}u$, viz. $\mathrm{D}.\mathrm{D}u$, we shall represent by D^2u; the total differential of D^2u, viz. $\mathrm{D}.\mathrm{D}^2u$, by D^3u, and so on; and the nth total differential of u will be represented by D^nu. And in accordance with our former notation, $\left(\frac{d^2u}{dx^2}\right)$, $\left(\frac{d^2u}{dy^2}\right)$ will severally represent the 2nd differential coefficients of u with respect to x and y, x and y being equicrescent variables; and $\left(\frac{d^2u}{dx\,dy}\right)$ will represent the 2nd differential coefficient of u,

formed by making first y and then x to vary. An analogous meaning attaches to such symbols, as

$$\left(\frac{d^n u}{dx^n}\right), \quad \left(\frac{d^n u}{dy^n}\right), \quad \left(\frac{d^{m+n} u}{dx^m\, dy^n}\right), \text{ \&c.}$$

Thus for instance by $\left(\frac{d^4 u}{dx^2\, dy^2}\right)$ is meant, the fourth derived-function of u, calculated by making y to vary twice, and subsequently x to vary twice: the order of factors in the denominators indicating the order in which the differentiations are performed. And here we must remark upon a defect in notation. If $u = \text{F}(x, y)$, for instance, the same symbol d^2u in the numerators of

$$\left(\frac{d^2u}{dx^2}\right), \quad \left(\frac{d^2u}{dx\,dy}\right), \quad \left(\frac{d^2u}{dy^2}\right),$$

means processes and results altogether different. In the first it means the second differential of u springing from two successive variations of x; in the second it represents d^2u springing from a variation of x, taking place on the back of a previous variation of y; and in the third, d^2u as originating in two successive variations of y. The brackets therefore are used in conjunction with the different denominators to indicate these variations of the same symbol, and are marks sufficiently distinctive to prevent confusion. But before we proceed further, we must prove the following proposition.

73.] If a function be differentiated many times in respect of independent variables which it contains, the result is the same, whatever be the order of the variables with respect to which it is differentiated: provided that it be differentiated the same number of times and with respect to the same variables.

Let $\quad u = \text{F}(x, y, z, \ldots\ldots)$

then $\quad \left(\frac{d^2u}{dx\,dy}\right) = \left(\frac{d^2u}{dy\,dx}\right).$

For the sake of convenience, using symbols of differentiation,

$$d_x \text{F}(x, y, z, \ldots) = \text{F}(x+dx, y, z, \ldots) - \text{F}(x, y, z, \ldots) \qquad (92)$$

$$d_y \text{F}(x, y, z, \ldots) = \text{F}(x, y+dy, z, \ldots) - \text{F}(x, y, z, \ldots) \qquad (93)$$

$\therefore$ from (92),

$$d_y d_x \text{F}(x, y, z, \ldots) = \text{F}(x+dx, y+dy, z, \ldots) - \text{F}(x+dx, y, z, \ldots) \\ - \text{F}(x, y+dy, z, \ldots) + \text{F}(x, y, z, \ldots);$$

and from (93),

$$d_x d_y \text{F}(x, y, z, \ldots) = \text{F}(x+dx, y+dy, z, \ldots) - \text{F}(x, y+dy, z, \ldots) \\ - \text{F}(x+dx, y, z, \ldots) + \text{F}(x, y, z, \ldots);$$

and as the two results are identical, it is manifest that

$$d_x d_y \text{F}(x, y, z, \ldots) = d_y d_x \text{F}(x, y, z, \ldots);$$

or, writing the result according to the notation of Art. 46,

$$\left(\frac{d^2 u}{dx\, dy}\right) = \left(\frac{d^2 u}{dy\, dx}\right).$$

As the proof here given does not depend on the differentiation having been performed with respect to two variables only, it is plain that an analogous theorem is true for a differentiation with respect to any number of variables; so that we may always interchange, in whatever manner it is convenient, the order in which the several differentiations are performed; as for instance:

$$d_x d_y d_z u = d_y d_z d_x u = d_z d_y d_x u;$$

$$\text{or} \quad \left(\frac{d^3 u}{dx\, dy\, dz}\right) = \left(\frac{d^3 u}{dy\, dz\, dx}\right) = \left(\frac{d^3 u}{dz\, dx\, dy}\right).$$

Hence also it follows, that if successive partial differentiations are performed on a function of many independent variables, by making x, y, and z to vary severally r times, s times, and t times, the order of these variations may be interchanged in any permutation, and the result is the same; thus

$$\text{if} \quad u = \text{F}(x, y, z, \ldots),$$

$$\frac{d^{r+s+t} u}{dx^r\, dy^s\, dz^t} = \frac{d^{s+t+r} u}{dy^s\, dz^t\, dx^r} = \frac{d^{t+r+s} u}{dz^t\, dx^r\, dy^s}$$

$$= \frac{d^{r+s+t} u}{dy\, dx^{r-1}\, dz^t\, dx\, dy^{s-1}} = \ldots\ldots \quad (90)$$

Of this property, thus proved in the general case, some particular examples are subjoined.

Ex. 1. $$u = \frac{x^2 - y^2}{x^2 + y^2};$$

$$\left(\frac{du}{dx}\right) = \frac{4xy^2}{(x^2+y^2)^2}, \qquad \left(\frac{du}{dy}\right) = \frac{-4x^2y}{(x^2+y^2)^2},$$

$$\left(\frac{d^2u}{dy\,dx}\right) = \frac{8xy(x^2-y^2)}{(x^2+y^2)^3}, \qquad \left(\frac{d^2u}{dx\,dy}\right) = \frac{8xy(x^2-y^2)}{(x^2+y^2)^3}.$$

Ex. 2. $$u = \frac{xy^2}{z^2 - a^2};$$

$$\left(\frac{du}{dx}\right) = \frac{y^2}{z^2-a^2}, \qquad \left(\frac{du}{dy}\right) = \frac{2xy}{z^2-a^2}, \qquad \left(\frac{du}{dz}\right) = \frac{-2xy^2z}{(z^2-a^2)^2},$$

$$\left(\frac{d^3u}{dx\,dy\,dz}\right) = \left(\frac{d^3u}{dy\,dx\,dz}\right) = \left(\frac{d^3u}{dz\,dy\,dx}\right) = \frac{-4yz}{(z^2-a^2)^2}.$$

74.] To differentiate a Function of two Independent variables.

Let $$u = \mathrm{F}(x, y);$$

$$\mathrm{D}u = \left(\frac{du}{dx}\right) dx + \left(\frac{du}{dy}\right) dy.$$

In general, $\left(\frac{du}{dx}\right)$ and $\left(\frac{du}{dy}\right)$ being functions of x and y,

$$\mathrm{D}\left(\frac{du}{dx}\right) = \left(\frac{d\left(\frac{du}{dx}\right)}{dx}\right) dx + \left(\frac{d\left(\frac{du}{dx}\right)}{dy}\right) dy,$$

$$= \left(\frac{d^2u}{dx^2}\right) dx + \left(\frac{d^2u}{dy\,dx}\right) dy;$$

$$\mathrm{D}\left(\frac{du}{dy}\right) = \left(\frac{d\left(\frac{du}{dy}\right)}{dx}\right) dx + \left(\frac{d\left(\frac{du}{dy}\right)}{dy}\right) dy,$$

$$= \left(\frac{d^2u}{dx\,dy}\right) dx + \left(\frac{d^2u}{dy^2}\right) dy;$$

whence, considering the most general case, when neither x nor y is equicrescent, we have

$$\mathrm{D}.\mathrm{D}u = \mathrm{D}\left(\frac{du}{dx}\right) dx + \mathrm{D}\left(\frac{du}{dy}\right) dy + \left(\frac{du}{dx}\right) d^2x + \left(\frac{du}{dy}\right) d^2y,$$

$$D^2u = \left(\frac{d^2u}{dx^2}\right) dx^2 + 2\left(\frac{d^2u}{dx\,dy}\right) dx\,dy + \left(\frac{d^2u}{dy^2}\right) dy^2$$
$$+ \left(\frac{du}{dx}\right) d^2x + \left(\frac{du}{dy}\right) d^2y; \qquad (100)$$

the brackets indicating that the derived-functions within them are partial. Similarly,

$$D^3u =$$
$$\left(\frac{d^3u}{dx^3}\right) dx^3 + 3\left(\frac{d^3u}{dx^2\,dy}\right) dx^2\,dy + 3\left(\frac{d^3u}{dx\,dy^2}\right) dx\,dy^2 + \left(\frac{d^3u}{dy^3}\right) dy^3$$
$$+3\left[\left(\frac{d^2u}{dx^2}\right) dx\,d^2x + \left(\frac{d^2u}{dx\,dy}\right)\{d^2x\,dy + d^2y\,dx\} + \left(\frac{d^2u}{dy^2}\right) dy\,d^2y\right]$$
$$+ \left(\frac{du}{dx}\right) d^3x + \left(\frac{du}{dy}\right) d^3y. \qquad (101)$$

Similarly may other total differentials be found; but the general term is too complicated to be of any practical use.

Now let the results be simplified by making x and y equicrescent, neither of which assumptions is inconsistent with the given equation; then

$$d^2x = d^3x = \ldots\ldots = 0; \qquad d^2y = d^3y = \ldots\ldots = 0;$$

and we have the following series of equations:

$$u = F(x, y),$$

$$Du = \left(\frac{du}{dx}\right) dx + \left(\frac{du}{dy}\right) dy,$$

$$D^2u = \left(\frac{d^2u}{dx^2}\right) dx^2 + 2\left(\frac{d^2u}{dx\,dy}\right) dx\,dy + \left(\frac{d^2u}{dy^2}\right) dy^2, \qquad (102)$$

$$D^3u = \left(\frac{d^3u}{dx^3}\right) dx^3 + 3\left(\frac{d^3u}{dx^2\,dy}\right) dx^2\,dy + 3\left(\frac{d^3u}{dx\,dy^2}\right) dx\,dy^2$$
$$+ \left(\frac{d^3u}{dy^3}\right) dy^3, \qquad (103)$$

and so on; the law of the coefficients being the same as that of $(1+x)^n$; which may be proved to be true for positive integral values of the exponent, by a train of reasoning similar to that in Art. 53; whence the nth differential is

$$\mathrm{D}^n u = \left(\frac{d^n u}{dx^n}\right) dx^n + n\left(\frac{d^n u}{dx^{n-1}\,dy}\right) dx^{n-1}\,dy$$

$$+ \frac{n(n-1)}{1.2}\left(\frac{d^n u}{dx^{n-2}\,dy^2}\right) dx^{n-2}\,dy^2 + \ldots + \left(\frac{d^n u}{dy^n}\right) dy^n. \quad (104)$$

Ex. 1. $u = (x^2+y^2)^{\frac{1}{2}}$;

$$\left(\frac{du}{dx}\right) = \frac{x}{(x^2+y^2)^{\frac{1}{2}}}, \quad \left(\frac{d^2u}{dx^2}\right) = \frac{y^2}{(x^2+y^2)^{\frac{3}{2}}}, \quad \left(\frac{d^3u}{dx^3}\right) = \frac{-3xy^2}{(x^2+y^2)^{\frac{5}{2}}},$$

$$\left(\frac{du}{dy}\right) = \frac{y}{(x^2+y^2)^{\frac{1}{2}}}, \quad \left(\frac{d^2u}{dx\,dy}\right) = \frac{-xy}{(x^2+y^2)^{\frac{3}{2}}}, \quad \left(\frac{d^3u}{dx^2\,dy}\right) = \frac{y(2x^2-y^2)}{(x^2+y^2)^{\frac{5}{2}}},$$

$$\left(\frac{d^2u}{dy^2}\right) = \frac{x^2}{(x^2+y^2)^{\frac{3}{2}}}, \quad \left(\frac{d^3u}{dx\,dy^2}\right) = \frac{x(2y^2-x^2)}{(x^2+y^2)^{\frac{5}{2}}},$$

$$\left(\frac{d^3u}{dy^3}\right) = \frac{-3yx^2}{(x^2+y^2)^{\frac{5}{2}}};$$

$$\mathrm{D}^3 u = \Big\{-3xy^2\,dx^3 + 3y(2x^2-y^2)\,dx^2\,dy$$

$$+ 3x(2y^2-x^2)\,dx\,dy^2 - 3yx^2\,dy^3\Big\}\frac{1}{(x^2+y^2)^{\frac{5}{2}}}.$$

Ex. 2. $u = e^{ax+by}$;

$$\left(\frac{du}{dx}\right) = a\,e^{ax+by}, \qquad \left(\frac{d^2u}{dx^2}\right) = a^2\,e^{ax+by},$$

$$\left(\frac{du}{dy}\right) = b\,e^{ax+by}, \qquad \left(\frac{d^2u}{dx\,dy}\right) = ab\,e^{ax+by},$$

$$\left(\frac{d^2u}{dy^2}\right) = b^2\,e^{ax+by};$$

$$\therefore\quad \mathrm{D}^2 u = \{a^2\,dx^2 + 2ab\,dx\,dy + b^2\,dy^2\}\,e^{ax+by},$$

$$= (a\,dx + b\,dy)^2\,e^{ax+by}.$$

75.] By a similar process we have the following series of equations, considering u to be a function of many independent variables, $x, y, z, \ldots\ldots$ all of which are equicrescent.

$$u = \mathrm{F}(x, y, z, \ldots\ldots)$$

$$\mathrm{D}u = \left(\frac{du}{dx}\right) dx + \left(\frac{du}{dy}\right) dy + \left(\frac{du}{dz}\right) dz + \ldots\ldots \quad (105)$$

$$\text{D}^2 u = \left(\frac{d^2u}{dx^2}\right) dx^2 + \left(\frac{d^2u}{dy^2}\right) dy^2 + \left(\frac{d^2u}{dz^2}\right) dz^2 + \ldots\ldots$$

$$+ 2\left(\frac{d^2u}{dy\,dz}\right) dy\,dz + 2\left(\frac{d^2u}{du\,dx}\right) du\,dx + 2\left(\frac{d^2u}{dx\,dy}\right) dx\,dy + \ldots (106)$$

. . . .

If $x, y, z \ldots\ldots$ are not equicrescent, terms will have to be added analogous to those which, by reason of x and y being equicrescent, vanished from equations (100) and (101).

Also, using the notation of partial differentials, the above equations become

$$\left.\begin{array}{l} \text{D}u = d_x u + d_y u + d_z u + \ldots\ldots\ldots \\ \text{D}^2 u = d^2{}_x u + d^2{}_y u + d^2{}_z u + 2d_y\,d_z u + 2d_z\,d_x u + 2d_x\,d_y u + \end{array}\right\} (107)$$

. . . .

Ex. 1. $u = \sin(ax + by + cz);$

$$\left(\frac{du}{dx}\right) = a\cos(ax+by+cz,), \quad \left(\frac{d^2u}{dx^2}\right) = -a^2\sin(ax+by+cz),$$

$$\left(\frac{du}{dy}\right) = b\cos(ax+by+cz), \quad \left(\frac{d^2u}{dy^2}\right) = -b^2\sin(ax+by+cz),$$

$$\left(\frac{du}{dz}\right) = c\cos(ax+by+cz), \quad \left(\frac{d^2u}{dz^2}\right) = -c^2\sin(ax+by+cz),$$

$$\left(\frac{d^2u}{dy\,dz}\right) = -bc\sin(ax+by+cz);$$

. . . .

$$\therefore \quad \text{D}^2 u = -(a^2 dx^2 + b^2 dy^2 + c^2 dz^2 + 2bc\,dy\,dz$$
$$+ 2ca\,dz\,dx + 2ab\,dx\,dy)\sin(ax + by + cz).$$

76.] By means of the general results of the last Article are proved certain properties of homogeneous functions due to Euler, and which are generally known by the name of Euler's Theorems of Homogeneous Functions.

DEF. A homogeneous function of many variables is one which has the sum of the indices of the variables in every term the same; and if the sum of the indices in each term $= n$, the function is said to be homogeneous of n dimensions.

Let $u = F(x, y, z, \ldots\ldots)$ be the homogeneous function of n dimensions and r variables; for $x, y, z, \ldots\ldots$ let $tx, ty, tz, \ldots\ldots$ be written, and suppose the function to become u' when these substitutions are made; then, by the definition of homogeneous functions,

$$u' = F(tx, ty, tz, \ldots\ldots) = t^n F(x, y, z, \ldots\ldots). \qquad (108)$$

$$\left.\begin{array}{llll} \text{Let} & tx = x', & \therefore & \dfrac{dx'}{dt} = x, \\ & ty = y', & & \dfrac{dy'}{dt} = y, \\ & tz = z', & & \dfrac{dz'}{dt} = z; \end{array}\right\} (109)$$

$$\therefore \quad u' = F(x', y', z', \ldots\ldots);$$

$$\therefore \quad \frac{Du'}{dt} = \left(\frac{dF}{dx'}\right)\frac{dx'}{dt} + \left(\frac{dF}{dy'}\right)\frac{dy'}{dt} + \left(\frac{dF}{dz'}\right)\frac{dz'}{dt} + \ldots\ldots$$

$$= x\left(\frac{dF}{dx'}\right) + y\left(\frac{dF}{dy'}\right) + z\left(\frac{dF}{dz'}\right) + \ldots\ldots \qquad (110)$$

and differentiating (108) with respect to t, we have

$$\frac{du'}{dt} = nt^{n-1} F(x, y, z, \ldots\ldots) \qquad (111)$$

equating which to (110), since they are equal,

$$nt^{n-1} F(x, y, z, \ldots\ldots) = x\left(\frac{dF'}{dx'}\right) + y\left(\frac{dF'}{dy'}\right) + z\left(\frac{dF}{dz'}\right) + \ldots$$

Let $t = 1$, then $x' = x$, $y' = y, \ldots\ldots$

$$\therefore \quad nF(x, y, z \ldots) = nu = x\left(\frac{dF}{dx}\right) + y\left(\frac{dF}{dy}\right) + z\left(\frac{dF}{dz}\right) + \ldots (112)$$

Again, taking the second total differential of u', and dividing by $(dt)^2$,

$$\frac{D^2u'}{dt^2} = \left(\frac{d^2u'}{dx'^2}\right)\frac{dx'^2}{dt^2} + 2\left(\frac{d^2u'}{dx'dy'}\right)\frac{dx'}{dt}\frac{dy'}{dt} + \left(\frac{d^2u'}{dy'^2}\right)\frac{dy'^2}{dt^2} + \ldots\ldots$$

and substituting from (109), we have

$$\frac{D^2u'}{dt^2} = \left(\frac{d^2u'}{dx'^2}\right)x^2 + 2\left(\frac{d^2u'}{dx'dy'}\right)xy + \left(\frac{d^2u'}{dy'^2}\right)y^2 + \ldots\ldots (113)$$

also differentiating (111) with respect to t, we have

$$\frac{d^2u'}{dt^2} = n(n-1)\, t^{n-2}\, \mathrm{F}(x, y, z, \ldots\ldots) \qquad (114)$$

whence, equating (113) and (114), and making $t = 1$, we have

$$n(n-1)\, u = x^2 \left(\frac{d^2u}{dx^2}\right) + 2\,xy \left(\frac{d^2u}{dy\, dy}\right) + y^2 \left(\frac{d^2u}{dy^2}\right) + \ldots (115)$$

Similarly may it be shewn that

$$n(n-1)\,(n-2)\, u = x^3 \left(\frac{d^3u}{dx^3}\right) + 3\,x^2y \left(\frac{d^3u}{dx^2 dy}\right)$$

$$+ 3\,xy^2 \left(\frac{d^3u}{dx\, dy^2}\right) + y^3 \left(\frac{d^3u}{dy^3}\right) + \ldots\ldots (116)$$

and similar theorems are true for any other order of differentials.

Ex. 1. $\mathrm{F}(x, y, z) = u = \mathrm{A}x^2 + \mathrm{B}y^2 + \mathrm{C}z^2 + \mathrm{E}yz + \mathrm{G}zx + \mathrm{H}xy$, which is an homogeneous function of two dimensions and three variables.

$$\left(\frac{du}{dx}\right) = 2\,\mathrm{A}x + \mathrm{G}z + \mathrm{H}y, \qquad \left(\frac{d^2u}{dx^2}\right) = 2\,\mathrm{A}, \qquad \left(\frac{d^2u}{dy\, dz}\right) = \mathrm{E},$$

$$\left(\frac{du}{dy}\right) = 2\,\mathrm{B}y + \mathrm{E}z + \mathrm{H}x, \qquad \left(\frac{d^2u}{dy^2}\right) = 2\,\mathrm{B}, \qquad \left(\frac{d^2u}{du\, dx}\right) = \mathrm{G},$$

$$\left(\frac{du}{dz}\right) = 2\,\mathrm{C}z + \mathrm{E}y + \mathrm{G}x, \qquad \left(\frac{d^2u}{dz^2}\right) = 2\,\mathrm{C}, \qquad \left(\frac{d^2u}{dx\, dy}\right) = \mathrm{H},$$

therefore by equations (112) and (115)

$$x\left(\frac{du}{dx}\right) + y\left(\frac{du}{dy}\right) + z\left(\frac{du}{dz}\right) = 2\{\mathrm{A}x^2 + \mathrm{B}y^2 + \mathrm{C}z^2 + \mathrm{E}yz + \mathrm{G}zx + \mathrm{H}xy\}$$

$$= 2u;$$

$$x^2 \left(\frac{d^2u}{dx^2}\right) + y^2 \left(\frac{d^2u}{dy^2}\right) + z^2 \left(\frac{d^2u}{dz^2}\right) + 2yz \left(\frac{d^2u}{dy\, dz}\right)$$

$$+ 2zx \left(\frac{d^2u}{dz\, dx}\right) + 2xy \left(\frac{d^2u}{dx\, dy}\right) =$$

$$2\,\{\mathrm{A}x^2 + \mathrm{B}y^2 + \mathrm{C}z^2 + \mathrm{E}yz + \mathrm{G}zx + \mathrm{H}xy\} = 2u.$$

Ex. 2. $u = \text{F}\left(\frac{y}{x}\right)$, which is a function of 0 dimensions;

$$\left(\frac{du}{dx}\right) = -\frac{y}{x^2}\text{F}'\left(\frac{y}{x}\right),$$

$$\left(\frac{du}{dy}\right) = \frac{1}{x}\text{F}'\left(\frac{y}{x}\right);$$

$$\therefore\ x\left(\frac{du}{dx}\right) + y\left(\frac{du}{dy}\right) = \left(\frac{y}{x} - \frac{y}{x}\right)\text{F}'\left(\frac{y}{x}\right) = 0.$$

77.] In the above discussion we have considered all the variables which enter into the function $\text{F}(x, y, z, \ldots\ldots)$ to be independent; but it may of course sometimes happen that two or more are functions of each other, or that all of them are functions of other variables; the principles involved in the previous Article are however sufficient for all such problems, and therefore it is unnecessary to consider them at length; but the student must be on his guard against an undue assumption of one or more equicrescent variables. There is however one case of great importance and wide application, which it is necessary to consider at some length, viz. that of an implicit function of two or more variables.

SECTION 6.—*Successive Differentiation of Implicit Functions.*

78.] Firstly, consider a function of two variables x and y, and of the form

$$u = \text{F}(x, y) = c,$$

c being a constant; then

$$\text{D}u = 0 = \left(\frac{du}{dx}\right)dx + \left(\frac{du}{dy}\right)dy, \tag{117}$$

$$\text{D}^2u = 0 = \left(\frac{d^2u}{dx^2}\right)dx^2 + 2\left(\frac{d^2u}{dx\,dy}\right)dx\,dy + \left(\frac{d^2u}{dy^2}\right)dy^2 + \left(\frac{du}{dx}\right)d^2x + \left(\frac{du}{dy}\right)d^2y, \tag{118}$$

$$\text{D}^3u = 0 = \left(\frac{d^3u}{dx^3}\right)dx^3 + \ldots\ldots$$

and considering x to be equicrescent, we have

$$\left(\frac{du}{dy}\right)\frac{dy}{dx} + \left(\frac{du}{dx}\right) = 0, \tag{119}$$

$$\left(\frac{du}{dy}\right)\frac{d^2y}{dx^2} + 2\left(\frac{d^2u}{dx\,dy}\right)\frac{dy}{dx} + \left(\frac{d^2u}{dy^2}\right)\frac{dy^2}{dx^2} + \left(\frac{d^2u}{dx^2}\right) = 0, \tag{120}$$

whence we have the following values of $\frac{dy}{dx}$, and of $\frac{d^2y}{dx^2}$:

$$\frac{dy}{dx} = -\frac{\left(\frac{du}{dx}\right)}{\left(\frac{du}{dy}\right)} \tag{121}$$

$$\frac{d^2y}{dx^2} = \frac{-\left(\frac{d^2u}{dx^2}\right)\left(\frac{du}{dy}\right)^2 + 2\left(\frac{d^2u}{dx\,dy}\right)\left(\frac{du}{dx}\right)\left(\frac{du}{dy}\right) - \left(\frac{d^2u}{dy^2}\right)\left(\frac{du}{dx}\right)^2}{\left(\frac{du}{dy}\right)^3} \tag{122}$$

The latter might have been derived from (121) by differentiating with respect to the equicrescent variable x, as well as from (120), whence we have deduced it.

Similarly from $\mathrm{D}^3 u = 0$ may be deduced $\frac{d^3y}{dx^3}$; and from subsequent total differentials the other derived-functions of y may be formed.

79.] Ex. 1. Given $y^2 - 2yx + a^2 = u = 0$, to find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

$$\left(\frac{du}{dx}\right) = -2y, \qquad \left(\frac{d^2u}{dx^2}\right) = 0,$$

$$\left(\frac{du}{dy}\right) = 2y-2x, \qquad \left(\frac{d^2u}{dx\,dy}\right) = -2,$$

$$\left(\frac{d^2u}{dy^2}\right) = 2;$$

$\therefore$ by (121) $$\frac{dy}{dx} = \frac{y}{y-x};$$

and by (122) $$\frac{d^2y}{dx^2} = \frac{y(y-2x)}{(y-x)^3}.$$

Instead however of introducing *formally* the general values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, given in equations (121) and (122), it is more convenient to differentiate the given function *immediately* according to the principles contained in Art. 48; the two following examples illustrate the process:

Ex. 2. $x^3+3axy+y^3=0$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Differentiating, we have

$$(x^2+ay)\,dx+(y^2+ax)\,dy=0; \tag{123}$$

$$\therefore \quad \frac{dy}{dx}=-\frac{x^2+ay}{y^2+ax};$$

$$\therefore \quad \frac{d^2y}{dx^2}=-\frac{\left(2x+a\frac{dy}{dx}\right)(y^2+ax)-\left(2y\frac{dy}{dx}+a\right)(x^2+ay)}{(y^2+ax)^2}$$

$$=\frac{2a^3xy}{(y^2+ax)^3}.$$

Ex. 3. $x^2+y^2=a^2$, to find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$,

$$x\,dx+y\,dy=0;$$

$$\therefore \quad y\frac{dy}{dx}+x=0, \qquad \text{and} \qquad \frac{dy}{dx}=-\frac{x}{y},$$

$$y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+1=0,$$

$$y\frac{d^2y}{dx^2}+\frac{x^2+y^2}{y^2}=0; \qquad \therefore \quad \frac{d^2y}{dx^2}=-\frac{a^2}{y^3}.$$

80.] Hence also, if an implicit function be given involving x and y, we may calculate the several coefficients of the powers of x in Maclaurin's Theorem (see Art. 55), and thus expand y in a series of terms of ascending powers of x.

Let the given implicit function be

$$u=\mathrm{F}(x,y)=c;$$

then, since in the series (13), Art. 55, $f(0)$, $f'(0)$, $f''(0)$, ...

are severally the values of y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, ... when $x=0$, we have by the equations (121) and (122):

$$y = f(x) = [y] + \left[-\frac{\left(\frac{du}{dx}\right)}{\left(\frac{du}{dy}\right)} \right] \frac{x}{1}$$

$$+ \left[\frac{-\left(\frac{d^2u}{dx^2}\right)\left(\frac{du}{dy}\right)^2 + 2\left(\frac{d^2u}{dx\,dy}\right)\left(\frac{du}{dx}\right)\left(\frac{du}{dy}\right) - \left(\frac{d^2u}{dy^2}\right)\left(\frac{du}{dx}\right)^2}{\left(\frac{du}{dy}\right)^3} \right] \frac{x^2}{1.2}$$

$$+ \cdot \cdot \cdot \cdot \cdot \qquad (124)$$

the square brackets being employed to signify that particular values of the coefficients are taken, viz. when $x = 0$.

Ex. 1. $ay^3 - xy = a$.

Let $x = 0$; $\therefore f(0) = 1$,

$$\frac{dy}{dx}(3ay^2 - x) - y = 0, \qquad f'(0) = \frac{1}{3a},$$

$$\frac{d^2y}{dx^2}(3ay^2 - x) + \frac{dy}{dx}\left(6ay\frac{dy}{dx} - 1\right) - \frac{dy}{dx} = 0, \quad f''(0) = 0,$$

$$\frac{d^3y}{dx^3}(3ay^2 - x) + 2\frac{d^2y}{dx^2}\left(6ay\frac{dy}{dx} - 1\right)$$

$$+ 6a\frac{dy}{dx}\left(\frac{dy^2}{dx^2} + y\frac{d^2y}{dx^2}\right) - \frac{d^2y}{dx^2} = 0, \qquad f'''(0) = -\frac{2}{27a^3},$$

. . . .

$$\therefore f(x) = y = 1 + \frac{1}{3a}\frac{x}{1} - \frac{2}{27a^3}\frac{x^3}{1.2.3} + \ldots\ldots$$

which is one of the values of y in terms of x deduced from the given cubic equation; the other two values may be found by taking the other two (impossible) cube roots of unity in the value of $f(0)$.

Ex. 2. $y^3 - 3y + x = 0$; $\therefore f(0) = 0$ and $= \pm\sqrt{3}$,

$$\frac{dy}{dx}(3y^2 - 3) + 1 = 0, \qquad f'(0) = \frac{1}{3} \text{ and } = -\frac{1}{6},$$

$$\frac{d^2y}{dx^2}(y^2-1)+2y\frac{dy^2}{dx^2} = 0, \qquad f''(0) = 0 \text{ and } = \mp\frac{\sqrt{3}}{36},$$

$$\frac{d^3y}{dx^3}(y^2-1)+6y\frac{dy}{dx}\frac{d^2y}{dx^2}+2\frac{dy^3}{dx^3} = 0,$$

$$f'''(0) = \frac{2}{27} \text{ and } = -\frac{1}{27},$$

. . . .

$$\therefore \quad y = \frac{x}{3}+\frac{2}{27}\frac{x^3}{1.2.3}+\ldots\ldots$$

$$y = \pm\sqrt{3}-\frac{x}{6}\mp\frac{\sqrt{3}}{36}\frac{x^2}{1.2}-\frac{1}{27}\frac{x^3}{1.2.3}+\ldots\ldots$$

and the three series give three different values of y, which are the three roots of the given cubic equation in terms of x.

Ex. 3. An equation also may often be put in different algebraic forms, and then expanded by Maclaurin's Theorem, and sometimes in a series of descending powers of x. As for instance, consider the last example

$$y^3 - 3y + x = 0.$$

Divide by x

$$\frac{y^3}{x} - \frac{3y}{x^{\frac{1}{3}}}\frac{1}{x^{\frac{2}{3}}} + 1 = 0.$$

For $\frac{y}{x^{\frac{1}{3}}}$ write y, and for $\frac{1}{x^{\frac{2}{3}}}$ write x; whereby we have

$$y^3 - 3yx + 1 = 0.$$

Expanding which, as in the last two examples, and taking only the possible cube root, we have

$$y = -1 - \frac{x}{1} + \frac{x^3}{3} - \ldots\ldots$$

and replacing y and x by their values, and multiplying through by $x^{\frac{1}{3}}$, we have

$$y = -x^{\frac{1}{3}} - \frac{1}{x^{\frac{1}{3}}} + \frac{1}{3x^{\frac{5}{3}}} + \ldots\ldots$$

81.] By the above process also may we expand a function which is of great importance in the calculation of many series, and thereby arrive at some numerical coefficients which are known by the name of *Bernoulli's Numbers*.

To expand
$$y = \frac{x}{e^x - 1} = f(x).$$

We may observe that $f(-x) = \dfrac{-x}{e^{-x}-1} = \dfrac{xe^x}{e^x-1}$;

$$\therefore\ f(x) - f(-x) = -x;$$

and as whatever odd power of x entered into the expansion of $f(x)$ would also enter into that of $f(x)-f(-x)$, it follows that no odd power of x enters into $f(x)$ except the first.

Now from above
$$y e^x = y + x; \tag{125}$$

$$\therefore\ e^x\left(\frac{dy}{dx}+y\right) = \frac{dy}{dx} + 1,$$

$$e^x\left(\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y\right) = \frac{d^2y}{dx^2},$$

$$e^x\left(\frac{d^3y}{dx^3}+3\frac{d^2y}{dx^2}+3\frac{dy}{dx}+y\right) = \frac{d^3y}{dx^3},$$

$$\cdots\cdots\cdots\cdots$$

$$e^x\left(\frac{d^ny}{dx^n}+n\frac{d^{n-1}y}{dx^{n-1}}+\ldots\ldots+n\frac{dy}{dx}+y\right) = \frac{d^ny}{dx^n}:$$

the last term being the general differential of (125). Let now $x = 0$ in these several equations, and we have

$f(0) = f(0)$, which is an identity;

$$f(0) = 1, \qquad \therefore\ f(0) = 1,$$

$$2f'(0)+f(0) = 0, \qquad f'(0) = -\frac{1}{2},$$

$$3f''(0)+3f'(0)+f(0) = 0, \qquad f''(0) = \frac{1}{6},$$

$$4f'''(0)+6f''(0)+4f'(0)+f(0) = 0, \qquad f'''(0) = 0,$$

$$5f''''(0)+10f'''(0)+10f''(0)+5f'(0)+f(0) = 0,$$
$$f''''(0) = -\frac{1}{30},$$

$$\cdots\cdots\cdots\cdots$$

$$nf^{n-1}(0) + \frac{n(n-1)}{1.2}f^{n-2}(0) + \ldots\ldots + nf'(0) + f(0) = 0. \tag{126}$$

the last term being the general equation, by means of which $f^{n-1}(0)$ may be determined in terms of the preceding coefficients. But the labour of calculating from this expression may be diminished by observing that, as $f(x)$ involves no odd powers of x except the first,

$$f'''(0) = \ldots\ldots = f^{2r-1}(0) = 0.$$

Hence in (126), if n be of the form $2r+1$, we have

$$(2r+1)f^{2r}(0) + \frac{(2r+1)\,2r\,(2r-1)}{1.2.3}f^{2r-2}(0) + \ldots\ldots$$
$$+ (2r+1)f'(0) + f(0) = 0, \qquad (127)$$

by means of which any coefficients may be found; and we have

$$\frac{x}{e^x-1} = 1 - \frac{1}{2}\frac{x}{1} + \frac{1}{6}\frac{x^2}{1.2} - \frac{1}{30}\frac{x^4}{1.2.3.4} + \frac{1}{42}\frac{x^6}{1.2.3.4.5.6} + \qquad (128)$$

The values of $f(0), f'(0), f''(0), \ldots\ldots$ are commonly called the numbers of Bernoulli; and though they do not *explicitly* follow any palpably regular law, yet they are implicitly connected with each other by the formula (127). It is convenient to represent them by distinctive symbols; we will therefore substitute as follows:

$$B_0 = 1,\quad B_1 = -\frac{1}{2},\quad B_2 = \frac{1}{6},\quad B_3 = 0,\quad B_4 = -\frac{1}{30},\ldots\ldots$$

Required to develope $\frac{1}{e^x+1}$ by Bernoulli's numbers.

$$\because\quad \frac{x}{e^x-1} - \frac{x}{e^x+1} = \frac{2x}{e^{2x}-1};$$

$$\therefore\quad \frac{x}{e^x+1} = \frac{x}{e^x-1} - \frac{2x}{e^{2x}-1},$$

$$= B_0 + B_1 x + B_2\frac{x^2}{1.2} + \ldots\ldots - \left\{B_0 + B_1\,2x + B_2\frac{4x^2}{1.2} + \ldots\right\}$$

$$\therefore\quad \frac{1}{e^x+1} = -B_1 - (2^2-1)B_2\frac{x}{1.2} - (2^4-1)B_4\frac{x^3}{1.2.3.4}$$
$$-(2^6-1)B_6\frac{x^5}{1.2.3.4.5.6}\ldots\ldots \qquad (129)$$

the law of which is sufficiently obvious.

To expand $\tan x$ by means of Bernoulli's numbers.

By equation (33), Art. 58,

$$\tan x = \frac{1}{\sqrt{-1}} \frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}},$$

$$= \frac{1}{\sqrt{-1}} \frac{e^{2x\sqrt{-1}} - 1}{e^{2x\sqrt{-1}} + 1},$$

$$= \frac{1}{\sqrt{-1}} \left\{1 - \frac{2}{e^{2x\sqrt{-1}} + 1}\right\};$$

whence, by means of equation (129),

$$\tan x = \frac{1}{\sqrt{-1}} \left\{1 + 2\mathrm{B}_1 + 2\sqrt{-1}\,(2^2-1)\,\mathrm{B}_2 \frac{2x}{1.2} \right.$$
$$\left. -2\sqrt{-1}\,(2^4-1)\,\mathrm{B}_4 \frac{2^3 x^3}{1.2.3.4} \cdots\cdots \right\}$$

$$= 2\left\{(2^2-1)\,\mathrm{B}_2 \frac{2x}{1.2} - (2^4-1)\,\mathrm{B}_4 \frac{2^3 x^3}{1.2.3.4} \right.$$
$$\left. + (2^6-1)\,\mathrm{B}_6 \frac{2^5 x^5}{1.2 \ldots 5.6} \cdots\cdots \right\} \quad (130)$$

which, being written at length, becomes

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots\cdots$$

and the law of the series is sufficiently obvious in equation (130). Also, differentiating the above, we can find an equation for $(\tan x)^2$.

Similarly may $\cot x$ and $(\cot x)^2$ be found.

82.] It is unnecessary to enter at any length on the *general* subject of implicit functions of more than two variables, as the principles above explained and illustrated are applicable to all such cases; but as particular forms occur, and particular examples have to be solved in the sequel of our work, it is convenient to consider them at this point of the Treatise where they naturally fall; such is

Lagrange's Theorem for the development of an implicit function of three variables, of the form $y = z + x\phi(y)$.

Given that $y = z + x\phi(y)$, in which equation y is an implicit function of two variables z and x, which are supposed to have no other relation to each other besides that given by this equation, so that they may vary independently of each other; it is required to determine in ascending powers of x another function of y, viz. $f(y)$.

Let $u = f(y)$, and therefore u is a function of x; whence, by Maclaurin's Series,

$$u = [u] + \left[\frac{du}{dx}\right]\frac{x}{1} + \left[\frac{d^2u}{dx^2}\right]\frac{x^2}{1.2} + \left[\frac{d^3u}{dx^3}\right]\frac{x^3}{1.2.3} + \ldots\ldots \quad (131)$$

bracketing the quantities, as in Art. 80, to indicate that particular values of them are to be taken, viz. when $x = 0$; that is, if $u = \mathrm{F}(x)$, $[u] = \mathrm{F}(0)$, $\left[\frac{du}{dx}\right] = \mathrm{F}'(0)$, and so on. Hence we have the following data:

$$u = f(y), \qquad y = z + x\phi(y); \quad (132)$$

$$\therefore \text{ when } x = 0, \; y = z,$$

$$\therefore \; [u] = f(z),$$

and calculating the partial derived-functions of (132), by considering y to vary in consequence of changes of x and of z, that is, calculating $\left(\frac{dy}{dx}\right)$ and $\left(\frac{dy}{dz}\right)$, we have

$$\left(\frac{dy}{dx}\right) = \phi(y) + x\phi'(y)\left(\frac{dy}{dx}\right), \qquad \therefore \; \left(\frac{dy}{dx}\right) = \frac{\phi(y)}{1 - x\phi'(y)};$$

$$\left(\frac{dy}{dz}\right) = 1 + x\phi'(y)\left(\frac{dy}{dz}\right), \qquad \therefore \; \left(\frac{dy}{dz}\right) = \frac{1}{1 - x\phi'(y)};$$

$$\therefore \; \left(\frac{dy}{dx}\right) = \phi(y)\left(\frac{dy}{dz}\right). \quad (133)$$

But $\left(\frac{du}{dx}\right) = \frac{du}{dy}\frac{dy}{dx}$;

whence, by reason of (133),

$$\frac{du}{dx} = \frac{du}{dy}\phi(y)\left(\frac{dy}{dz}\right).$$

Let $x = 0$, then $y = z$; $\therefore$ $dy = dz$, and $u = [u] = f(z)$;

$$\therefore \quad \left[\frac{du}{dx}\right] = \frac{d.f(z)}{dz}\phi(z). \tag{134}$$

Again, as
$$\frac{du}{dx} = \frac{du}{dy}\phi(y)\frac{dy}{dz},$$

and as $\frac{du}{dy}$ and $\phi(y)$ are explicitly functions of y only, (though they are also functions of x and z implicitly by virtue of equation 132,) and $\frac{dy}{dz}$ is explicitly a function of x and z, it is convenient for the purposes of differentiation to consider $\frac{du}{dx}$ the product of *two* functions, viz. of $\frac{du}{dy}\phi(y)$ and of $\frac{dy}{dz}$; whence, differentiating, we have

$$\begin{aligned}
\frac{d^2u}{dx^2} &= \frac{d}{dx}\left\{\frac{du}{dy}\phi(y)\frac{dy}{dz}\right\}, \\
&= \frac{d\left\{\frac{du}{dy}\phi(y)\right\}}{dy}\frac{dy}{dx}\frac{dy}{dz} + \frac{du}{dy}\phi(y)\frac{d^2y}{dx\,dz}, \\
&= \frac{d\left\{\frac{du}{dy}\phi(y)\right\}}{dy}\frac{dy}{dz}\frac{dy}{dx} + \frac{du}{dy}\phi(y)\frac{d^2y}{dz\,dx}, \\
&= \frac{d}{dz}\left\{\frac{du}{dy}\phi(y)\frac{dy}{dx}\right\},
\end{aligned}$$

and substituting for $\frac{dy}{dx}$ from equation (133),

$$\frac{d^2u}{dx^2} = \frac{d}{dz}\left\{\frac{du}{dy}\{\phi(y)\}^2\frac{dy}{dz}\right\}.$$

Let $x = 0$, in which case, as before, $y = z$, $dy = dz$, and $u = f(z)$;

$$\therefore \quad \left[\frac{d^2u}{dx^2}\right] = \frac{d}{dz}\left\{\frac{d.f(z)}{dz}\{\phi(z)\}^2\right\}.$$

Again, considering $\frac{d^2u}{dx^2}$ to involve a product of two functions, viz. $\frac{du}{dy}\{\phi(y)\}^2$ and $\frac{dy}{dz}$, the former of which is explicitly a

function of y only, and the latter is an explicit function of both x and y, and differentiating and substituting from (133),

$$\frac{d^3u}{dx^3} = \frac{d^2}{dz^2}\left\{\frac{du}{dy}\{\phi(y)\}^3\frac{dy}{dz}\right\};$$

$$\therefore \quad \left[\frac{d^3u}{dx^3}\right] = \frac{d^2}{dz^2}\left\{\frac{d.f(z)}{dz}\{\phi(z)\}^3\right\};$$

Let us then assume that the form is true for $\frac{d^{n-1}u}{dx^{n-1}}$; viz.:

$$\frac{d^{n-1}u}{dx^{n-1}} = \frac{d^{n-2}}{dz^{n-2}}\left\{\frac{du}{dy}\{\phi(y)\}^{n-1}\frac{dy}{dz}\right\},$$

$$\therefore \quad \frac{d^nu}{dx^n} = \frac{d}{dx}\frac{d^{n-2}}{dz^{n-2}}\left\{\frac{du}{dy}\{\phi(y)\}^{n-1}\frac{dy}{dz}\right\};$$

and since the order of differentiation may be reversed by virtue of Art. 73,

$$\frac{d^nu}{dx^n} = \frac{d^{n-2}}{dz^{n-2}}\frac{d}{dx}\left\{\frac{du}{dy}\{\phi(y)\}^{n-1}\frac{dy}{dz}\right\},$$

$$= \frac{d^{n-2}}{dz^{n-2}}\left\{\frac{d\left\{\frac{du}{dy}\{\phi(y)\}^{n-1}\right\}}{dy}\frac{dy}{dz}\frac{dy}{dx} + \frac{du}{dy}\{\phi(y)\}^{n-1}\frac{d^2y}{dz\,dx}\right\}$$

$$= \frac{d^{n-2}}{dz^{n-2}}\left\{\frac{d\left\{\frac{du}{dy}\{\phi(y)\}^{n-1}\right\}}{dy}\frac{dy}{dx}\frac{dy}{dz} + \frac{du}{dy}\{\phi(z)\}^{n-1}\frac{d^2y}{dx\,dz}\right\}$$

$$= \frac{d^{n-2}}{dz^{n-2}}\frac{d}{dz}\left\{\frac{du}{dy}\{\phi(y)\}^{n-1}\frac{dy}{dx}\right\},$$

$$= \frac{d^{n-1}}{dz^{n-1}}\left\{\frac{du}{dy}\{\phi(y)\}^{n}\frac{dy}{dz}\right\}, \text{ by virtue of equation (133)};$$

$$\therefore \quad \left[\frac{d^nu}{dx^n}\right] = \frac{d^{n-1}}{dz^{n-1}}\left\{\frac{d.f(z)}{dz}\{\phi(z)\}^n\right\}. \qquad (135)$$

If therefore the formulæ are true for $n - 1$, they are true for n; they are true when $n = 3$, therefore they are true when $n = 4$, and therefore are true for all positive integral values of n, which are the only cases in which it is necessary for us to find them. Substituting then, in equation (131), the values above determined, we have

$$f(y)=f(z)+\frac{d.f(z)}{dz}\phi(z)\frac{x}{1}+\frac{d}{dz}\left\{\frac{d.f(z)}{dz}\{\phi(z)\}^2\right\}\frac{x^2}{1.2}$$

$$+\frac{d^2}{dz^2}\left\{\frac{d.f(z)}{dz}\{\phi(z)\}^3\right\}\frac{x^3}{1.2.3}+\ldots\ldots$$

$$\ldots\ldots+\frac{d^{n-1}}{dz^{n-1}}\left\{\frac{d.f(z)}{dz}\{\phi(z)\}^n\right\}\frac{x^n}{1.2.3\ldots n}+\ldots\ldots \quad (136)$$

If, having given $y = z + x\phi(y)$, the problem were to determine y, then $f(y) = y$, and $f(z) = z$; and the above formula becomes

$$y=z+\phi(z)\frac{x}{1}+\frac{d}{dz}\left\{\phi(z)\right\}^2\frac{x^2}{1.2}+\frac{d^2}{dz^2}\left\{\phi(z)\right\}^3\frac{x^3}{1.2.3}+\ldots(137)$$

In applying the above theorems to particular examples, it is most convenient first to substitute the specific forms of the functions, and subsequently to replace the variables by their specific values.

Ex. 1. Given $a - by + cy^2 = 0$; to find y.

On comparing the given equation $y = \frac{a}{b} + \frac{c}{b}y^2$ with the typical form $y = z + x\phi(y)$, we have

$$\left.\begin{aligned} f(y) &= y \\ \therefore\ f(z) &= z \end{aligned}\right\} \quad \left.\begin{aligned} \phi(y) &= y^2 \\ \phi(z) &= z^2 \end{aligned}\right\} \quad \left.\begin{aligned} z &= \frac{a}{b} \\ x &= \frac{c}{b} \end{aligned}\right\};$$

whence we have

$$f(y)=f(z)+\frac{d.f(z)}{dz}\phi(z)\frac{x}{1}+\frac{d}{dz}\left\{\frac{d.f(z)}{dz}\{\phi(z)\}^2\right\}\frac{x^2}{1.2}$$

$$+\frac{d^2}{dz^2}\left\{\frac{d.f(z)}{dz}\{\phi(z)\}^3\right\}\frac{x^3}{1.2.3}+\ldots\ldots$$

$$y=z+z^2\frac{x}{1}+\frac{d}{dz}\left\{z^4\right\}\frac{x^2}{1.2}+\frac{d^2}{dz^2}\left\{z^6\right\}\frac{x^3}{1.2.3}+\ldots\ldots$$

$$=\ .\ .\ .\ .\ .\ .\ .\ +4z^3\frac{x^2}{1.2}+6.5.z^4\frac{x^3}{1.2.3}+\ldots\ldots$$

$$\therefore \quad y = \frac{a}{b} + \frac{a^2}{b^2}\frac{c}{b} + 4\frac{a^3}{b^3}\frac{c^2}{b^2}\frac{1}{1.2} + 6.5\frac{a^4}{b^4}\frac{c^3}{b^3}\frac{1}{1.2.3} + \dots\dots$$

$$= \frac{a}{b}\left\{1 + \frac{ac}{b^2} + \frac{4}{1.2}\frac{a^2c^2}{b^4} + \frac{6.5}{1.2.3}\frac{a^3c^3}{b^6} + \dots\dots\right\};$$

a series which is identical with that arising from the development of $\frac{b}{2c} - \frac{(b^2-4ac)^{\frac{1}{2}}}{2c}$, which is the least of the two roots of the given equation.

Ex. 2. Given $y^3 - ay + b = 0$, to find y^n.

On comparing the given equation $y = \frac{b}{a} + \frac{1}{a}y^3$ with the typical form, we have

$$\left.\begin{aligned} f(y) &= y^n \\ \therefore\ f(z) &= z^n \end{aligned}\right\} \quad \left.\begin{aligned} \phi(y) &= y^3 \\ \phi(z) &= z^3 \end{aligned}\right\} \quad \left.\begin{aligned} z &= \frac{b}{a} \\ x &= \frac{1}{a} \end{aligned}\right\};$$

$$\therefore \quad f(y) = f(z) + \frac{d.f(z)}{dz}\phi(z)\frac{x}{1} + \frac{d}{dz}\left\{\frac{d.f(z)}{dz}\{\phi(z)\}^2\right\}\frac{x^2}{1.2}$$

$$+ \frac{d^2}{dz^2}\left\{\frac{d.f(z)}{dz}\{\phi(z)\}^3\right\}\frac{x^3}{1.2.3} + \dots\dots$$

$$y^n = z^n + nz^{n-1}z^3\frac{x}{1} + \frac{d}{dz}\left\{nz^{n-1}z^6\right\}\frac{x^2}{1.2}$$

$$+ \frac{d^2}{dz^2}\left\{nz^{n-1}z^9\right\}\frac{x^3}{1.2.3} + \dots\dots$$

$$= \dots + nz^{n+2}\frac{x}{1} + n(n+5)z^{n+4}\frac{x^2}{1.2}$$

$$+ n(n+8)(n+7)z^{n+6}\frac{x^3}{1.2.3} + \dots\dots$$

$$y^n = \frac{b^n}{a^n}\left\{1 + n\frac{b^2}{a^2}\frac{1}{a} + \frac{n(n+5)}{1.2}\frac{b^4}{a^4}\frac{1}{a^2}\right.$$

$$\left. + \frac{n(n+8)(n+7)}{1.2.3}\frac{b^6}{a^6}\frac{1}{a^3} + \dots\dots\right\}.$$

Ex. 3. Given $y = a + be^y$; to find $\log_e y$.

$$\left.\begin{aligned} f(y) &= \log_e y \\ f(z) &= \log_e z \end{aligned}\right\} \quad \left.\begin{aligned} \phi(y) &= e^y \\ \phi(z) &= e^z \end{aligned}\right\} \quad \left.\begin{aligned} z &= a \\ x &= b \end{aligned}\right\};$$

$$f(y) = f(z) + \frac{d.f(z)}{dz}\phi(z)\frac{x}{1} + \frac{d}{dz}\left\{\frac{d.f(z)}{dz}\{\phi(z)\}^2\right\}\frac{x^2}{1.2}$$

$$+ \frac{d^2}{dz^2}\left\{\frac{d.f(z)}{dz}\{\phi(z)\}^3\right\}\frac{x^3}{1.2.3} + \ldots\ldots$$

$$\log_e y = \log_e z + \frac{e^z}{z}\frac{x}{1} + \frac{d}{dz}\left\{\frac{e^{2z}}{z}\right\}\frac{x^2}{1.2} + \frac{d^2}{dz^2}\left\{\frac{e^{3z}}{z}\right\}\frac{x^3}{1.2.3} + \ldots$$

$$= \;.\;\;.\;\;.\; + \frac{2z-1}{z^2}e^{2z}\frac{x^2}{1.2} + \frac{9z^2-6z+2}{z^3}e^{3z}\frac{x^3}{1.2.3} + \ldots$$

$$= \log_e a + \frac{e^a}{a}\frac{b}{1} + \frac{2a-1}{a^2}e^{2a}\frac{b^2}{1.2}$$

$$+ \frac{9a^2-6a+2}{a^3}e^{3a}\frac{b^3}{1.2.3} + \ldots\ldots$$

Ex. 4. Given $y = a + e\sin y$; to find $\cos y$ and $\sin 2y$.

(a) To find $\cos y$.

$$\left.\begin{aligned} f(y) &= \cos y \\ f(z) &= \cos z \end{aligned}\right\} \quad \left.\begin{aligned} \phi(y) &= \sin y \\ \phi(z) &= \sin z \end{aligned}\right\} \quad \left.\begin{aligned} x &= e \\ z &= a \end{aligned}\right\};$$

$$\cos y = \cos z - (\sin z)^2\frac{x}{1} - \frac{d}{dz}\left\{(\sin z)^3\right\}\frac{x^2}{1.2}$$

$$- \frac{d^2}{dz^2}\left\{(\sin z)^4\right\}\frac{x^3}{1.2.3} + \ldots\ldots$$

$$= \;.\;\;.\; - 3(\sin z)^2\cos z\frac{x^2}{1.2}$$

$$+ 4\{4(\sin z)^4 - 3(\sin z)^2\}\frac{x^3}{1.2.3} + \ldots\ldots$$

$$\cos y = \cos a - (\sin a)^2\frac{e}{1} - 3(\sin a)^2\cos a\frac{e^2}{1.2}$$

$$+ 4(\sin a)^2\{4(\sin a)^2 - 3\}\frac{e^3}{1.2.3} + \ldots\ldots$$

(β) To find $\sin 2y$.

$$\left.\begin{array}{l} f(y) = \sin 2y \\ f(z) = \sin 2z \end{array}\right\} \quad \left.\begin{array}{l} \phi(y) = \sin y \\ \phi(z) = \sin z \end{array}\right\} \quad \left.\begin{array}{l} x = e \\ z = a \end{array}\right\};$$

$$\sin 2y = \sin 2z + 2\cos 2z \sin z \frac{x}{1} + \frac{d}{dz}\left\{2\cos 2z (\sin z)^2\right\}\frac{x^2}{1.2} + \dots$$

$$= . \quad . \quad +4\cos 3z \sin z \frac{x^2}{1.2} + \dots\dots$$

$$= \sin 2a + 2\cos 2a \sin a \frac{e}{1} + 4\cos 3a \sin a \frac{e^2}{1.2} + \dots\dots$$

83.] A more general form of expansion than that explained and illustrated in the last two Articles was discovered by Laplace, and is known by the name of Laplace's Theorem.

Given $\quad y = \text{F}\{z + x\phi(y)\}$; to find $f(y)$.

Using the same notation of Maclaurin's Theorem as heretofore, we have

$$u = [u] + \left[\frac{du}{dx}\right]\frac{x}{1} + \left[\frac{d^2u}{dx^2}\right]\frac{x^2}{1.2} + \left[\frac{d^3u}{dx^3}\right]\frac{x^3}{1.2.3} + \dots\dots$$

Let $\quad u = f(y); \qquad \therefore \quad [u] = f\{\text{F}(z)\}.$

By Example 4, Art. 51,

$$\left(\frac{dy}{dx}\right) = \phi(y)\left(\frac{dy}{dz}\right) \tag{138}$$

$$\therefore \quad \frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} = \frac{du}{dy}\phi(y)\frac{dy}{dz}.$$

Let $x = 0$, $\qquad$ then $y = \text{F}(z)$, $\qquad u = f\{\text{F}(z)\}$;

$$\therefore \quad \left[\frac{du}{dx}\right] = \frac{d.f\{\text{F}(z)\}}{dz}\phi\{\text{F}(z)\}.$$

And by a process similar to that employed in the proof of Lagrange's Theorem, it may be shewn that

$$\left[\frac{d^2u}{dx^2}\right] = \frac{d}{dz}\left\{\frac{d.f\{\text{F}(z)\}}{dz}(\phi\{\text{F}(z)\})^2\right\},$$

and so on for other and for the nth terms; whence

$$f(y) = f\{\mathrm{F}(z)\} + \frac{d.f\{\mathrm{F}(z)\}}{dz}\,\phi\{\mathrm{F}(z)\}\,\frac{x}{1}$$

$$+\frac{d}{dz}\left\{\frac{d.f\{\mathrm{F}(z)\}}{dz}\,(\phi\{\mathrm{F}(z)\})^2\right\}\frac{x^2}{1.2} + \ldots\ldots \quad (139)$$

As an example of this Theorem, let it be required to find e^y, having given

$$y = \log\{z + x\sin y\};$$

$$f(y) = e^y, \qquad \mathrm{F}(z) = \log z, \qquad \phi(y) = \sin y,$$

$$f(z) = e^z; \qquad \therefore\ f.\mathrm{F}(z) = z; \qquad \therefore\ \phi\{\mathrm{F}(z)\} = \sin\log z;$$

whence, substituting in the formula (139), we have

$$e^y = z + \sin\log z\,\frac{x}{1} + \frac{d}{dz}\left\{(\sin\log z)^2\right\}\frac{x^2}{1.2}$$

$$+\frac{d^2}{dz^2}\left\{(\sin\log z)^3\right\}\frac{x^3}{1.2.3} + \ldots\ldots$$

$$= \ldots\ldots + \frac{2\sin\log z\ \cos\log z}{z}\,\frac{x^2}{1.2} + \ldots\ldots$$

$$= z + \sin\log z\,\frac{x}{1} + \frac{\sin(2\log z)}{z}\,\frac{x^2}{1.2} + \ldots\ldots$$

$$= \ldots\ldots + \frac{\sin(\log z^2)}{z}\,\frac{x^2}{1.2} + \ldots\ldots$$

Taylor's Series, it may be observed, is only a particular case of Laplace's; for as

$$y = \mathrm{F}\{z + x\phi(y)\}, \qquad \text{let } \phi(y) = a, \qquad \text{and } f(y) = y;$$

$$\therefore\ y = \mathrm{F}(z+ax) = \mathrm{F}(z) + \frac{d.\mathrm{F}(z)}{dz}\,\frac{ax}{1} + \frac{d^2.\mathrm{F}(z)}{dz^2}\,\frac{a^2x^2}{1.2} + \ldots\ldots$$

and writing h for ax, we have

$$\mathrm{F}(z+h) = \mathrm{F}(z) + \frac{d.\mathrm{F}(z)}{dz}\,\frac{h}{1} + \frac{d^2.\mathrm{F}(z)}{dz^2}\,\frac{h^2}{1.2} + \frac{d^3.\mathrm{F}(z)}{dz^3}\,\frac{h^3}{1.2.3} + \ldots$$

84.] We shall conclude this section of expansions with the following extension of Maclaurin's Theorem; the method of which gives rise to that process of Derivation, as it is called*, on which Arbogast has constructed his Calcul des Dérivations.

* This process, though called by the same name, is essentially different from that explained in Art. 18. The title of Arbogast's work is, Du Calcul des Dérivations; it was published at Strasbourg, An VIII. (1800).

Let $y = f(z)$, where

$$z = a_0 + a_1 \frac{x}{1} + a_2 \frac{x^2}{1.2} + a_3 \frac{x^3}{1.2.3} + \dots\dots \tag{140}$$

$$\therefore \quad \frac{dz}{dx} = a_1 + a_2 \frac{x}{1} + a_3 \frac{x^2}{1.2} + \dots\dots$$

$$\frac{d^2z}{dx^2} = a_2 + a_3 \frac{x}{1} + a_4 \frac{x^2}{1.2} + \dots\dots$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\therefore \text{ when } x = 0, \qquad z = a_0, \qquad \frac{dz}{dx} = a_1, \qquad \frac{d^2z}{dx^2} = a_2 \dots$$

In whatever manner therefore the general value of $\frac{d^ny}{dx^n}$ is composed of $f(z)$, $f'(z)$, $f''(z)$, ... combined with $\frac{dz}{dx}$, $\frac{d^2z}{dx^2}$, ... in the same manner will its value, when $x = 0$, be composed of $f(a_0)$, $f'(a_0)$, $f''(a_0)$, ... combined with a_1, a_2, a_3 ...... Hence it is plain that the value of $\frac{d^ny}{dx^n}$, when $x = 0$, may be obtained from the nth *differential* of $f(a_0)$ on the supposition that $da_0 = a_1$, $da_1 = a_2$, $da_2 = a_3$, ... $da_m = a_{m+1}$; and in this power of substitution consists the Method of Derivations.

Replacing therefore the successive coefficients of the powers of x in Maclaurin's Series, equation (13), Art. 55, by their values determined as above, we have

$$y = f(a_0) + [d.f(a_0)] \frac{x}{1} + [d^2.f(a_0)] \frac{x^2}{1.2} + \dots\dots \tag{141}$$

the square brackets indicating that particular values of the functions enclosed in them are to be taken, viz. when we replace da_0 by a_1, da_1 by a_2, ...

Ex. 1. It is required to expand $y = \frac{1}{z}$, where

$$z = a_0 + a_1 \frac{x}{1} + a_2 \frac{x^2}{1.2} + \dots\dots$$

$$\therefore \quad f(a_0) = \frac{1}{a_0};$$

$$[d.f(a_0)] = \left[-\frac{da_0}{a_0^2}\right] = -\frac{a_1}{a_0^2},$$

$$[d^2 f(a_0)] = \left[-\frac{(a_0)^2 da_1 - 2a_0 a_1 da_0}{a_0^4}\right] = \frac{2a_1^2 - a_0 a_2}{a_0^3},$$

$$[d^3 f(a_0)] = \frac{6a_0 a_1 a_2 - a_0^2 a_3 - 6a_1^3}{a_0^4},$$

. . . .

$$\therefore\ y = \frac{1}{a_0 + a_1 \frac{x}{1} + a_2 \frac{x^2}{1.2} + \dots\dots}$$

$$= \frac{1}{a_0} - \frac{a_1}{a_0^2} x + \frac{2a_1^2 - a_0 a_2}{a_0^3} \frac{x^2}{1.2}$$

$$+ \frac{6a_0 a_1 a_2 - a_0^2 a_3 - 6a_1^3}{a_0^4} \frac{x^3}{1.2.3} + \dots\dots$$

Ex. 2. To develope $y = z^m$, where

$$z = a_0 + a_1 x + a_2 \frac{x^2}{1.2} + \dots\dots$$

$$\therefore\ f(a_0) = a_0^m;$$

$$[d.f(a_0)] = [ma_0^{m-1} da_0] = ma_0^{m-1} a_1,$$

$$[d^2 f(a_0)] = [m(m-1) a_0^{m-2} a_1 da_0 + ma_0^{m-1} da_1]$$

$$= m(m-1) a_0^{m-2} a_1^2 + ma_0^{m-1} a_2,$$

$$[d^3 f(a_0)] = m(m-1)(m-2) a_0^{m-3} a_1^3$$

$$+ 3m(m-1) a_0^{m-2} a_1 a_2 + ma_0^{m-1} a_3,$$

. . . .

$$\therefore\ \left\{a_0 + a_1 \frac{x}{1} + a_2 \frac{x^2}{1.2} + \dots\dots\right\}^m =$$

$$a_0^m + ma_0^{m-1} a_1 \frac{x}{1} + ma_0^{m-2} \{(m-1) a_1^2 + a_0 a_2\} \frac{x^2}{1.2}$$

$$+ \{m(m-1)(m-2) a_0^{m-3} a_1^3 + 3m(m-1) a_0^{m-2} a_1 a_2$$

$$+ ma_0^{m-1} a_3\} \frac{x^3}{1.2.3} + \dots\dots$$

which is the Multinomial Theorem; and in which it is to be observed, that a_r does not appear before the coefficient of x^r.

Ex. 3. To develope $y = e^z$, where

$$z = a_0 + a_1 x + a_2 \frac{x^2}{1.2} + \dots\dots$$

$$\therefore\ f(a_0) = e^{a_0},$$

$$[d.f(a_0)] = [e^{a_0} d_{a_0}] = a_1 e^{a_0},$$

$$[d^2 f(a_0)] = [e^{a_0} a_1 da_0 + e^{a_0} da_1] = e^{a_0} a_1^2 + e^{a_0} a_2,$$

$$[d^3 f(a_0)] = e^{a_0} \{a_1^3 + 3a_1 a_2 + a_3\},$$

. . . .

$$\therefore \quad e^{(a_0 + a_1 x + \ldots\ldots)} = e^{a_0} \{1 + a_1 x + (a_1^2 + a_2) \frac{x^2}{1.2} + \ldots\ldots\}$$

Similarly may we expand $\sin(a_0 + a_1 x + a_2 \frac{x^2}{1.2} + \ldots\ldots)$ and $\log_e(a_0 + a_1 x + \ldots\ldots)$, the latter of which is useful in finding the sums of the powers of the roots of an equation.

Sometimes, instead of finding the actual coefficients, it is desirable to find the law of their dependence, that is, to determine the equation by means of which they are related to each other; such as has been found by implicit differentiation in Art. 81, equation (126).

Thus in Ex. 1 above,

$$\text{Suppose} \quad y = \frac{1}{z} = A_0 + A_1 x + A_2 \frac{x^2}{1.2} + \ldots\ldots \tag{142}$$

$$z = a_0 + a_1 x + a_2 \frac{x^2}{1.2} + \ldots\ldots \tag{143}$$

$$\text{then} \quad zy = 1; \tag{144}$$

$$\text{and} \quad zy = a_0 A_0 + C_1 \frac{x}{1} + C_2 \frac{x^2}{1.2} + C_3 \frac{x^3}{1.2.3} + \ldots\ldots \tag{145}$$

where $C_1, C_2, C_3, \ldots\ldots$ are the coefficients of the powers of x, arising from the product of the developed values of y and z, and which, as determined above, are the several values of $d.A_0 a_0$, $d^2.A_0 a_0$, $d^3.A_0 a_0, \ldots\ldots$ on the supposition that $da_0 = a_1$, $da_1 = a_2, \ldots\ldots dA_0 = A_1$, $dA_1 = A_2$, $dA_2 = A_3, \ldots\ldots$; but, on comparing (144) and (145), it follows, that

$$A_0 a_0 = 1;$$

$$\therefore \quad d^n A_0 a_0 = 0.$$

And therefore by Leibnitz's Theorem, Art. 53, equation (5),

$$a_0 d^n A_0 + n\, da_0\, d^{n-1} A_0 + \frac{n(n-1)}{1.2} d^2 a_0\, d^{n-2} A_0 + \ldots\ldots$$
$$+ n\, dA_0\, d^{n-1} a_0 + A_0 d^n a_0 = 0;$$

and therefore

$$A_n a_0 + n A_{n-1} a_1 + \frac{n(n-1)}{1.2} A_{n-2} a_2 + \ldots\ldots + A_0 a_n = 0; \tag{146}$$

whence $A_1, A_2, \ldots\ldots$ may be successively calculated.

As an example of this process, consider the problem which was discussed in Art. 81, viz. the expansion of

$$y = \frac{x}{e^x - 1},$$

$$= \frac{1}{1 + \frac{1}{2}\frac{x}{1} + \frac{1}{3}\frac{x^2}{1.2} + \frac{1}{4}\frac{x^3}{1.2.3} + \ldots\ldots}$$

Here we have merely to replace in (146) $a_0, a_1, a_2, \ldots\ldots$ severally by $1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots$ and we have

$$A_n + \frac{1}{2} n A_{n-1} + \frac{1}{3}\frac{n(n-1)}{1.2} A_{n-2} + \ldots\ldots + \frac{1}{n+1} A_0 = 0;$$

by means of which the successive coefficients can easily be calculated.

Thus let $n = 1$;

$$\therefore \quad A_1 + \frac{1}{2} A_0 = 0, \qquad \text{but } A_0 = \frac{1}{a_0} = 1;$$

$$\therefore \quad A_1 = -\frac{1}{2}.$$

$n = 2$,

$$A_2 + \frac{1}{2} A_1 + \frac{1}{3} A_0 = 0, \qquad \therefore \quad A_2 = \frac{1}{6},$$

and by a similar process may the values of $A_3, A_4, \ldots\ldots$ be found; and these are severally the numbers of Bernoulli.

Again, if $u = f(\alpha, \beta, \gamma, \ldots\ldots)$, where

$$\alpha = a_0 + a_1 x + a_2 \frac{x^2}{1.2} + \ldots\ldots$$

$$\beta = b_0 + b_1 x + b_2 \frac{x^2}{1.2} + \ldots\ldots$$

$$\gamma = c_0 + c_1 x + c_2 \frac{x^2}{1.2} + \ldots\ldots$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

then $\qquad u = A_0 + A_1 x + A_2 \frac{x^2}{1.2} + \ldots\ldots$

where $\qquad A_0 = f(a_0, b_0, c_0, \ldots\ldots)$

and $\qquad A_n = d^n f(a_0, b_0, c_0 \ldots\ldots)$, the total differentials being taken, and on the supposition, as heretofore, of $da_0 = a_1$, $da_1 = a_2, \ldots\ldots, db_0 = b_1,\ db_1 = b_2, \ldots\ldots, dc_0 = c_1,\ dc_1 = c_2 \ldots.$

For example:

$$\frac{a_0 + a_1 \frac{x}{1} + a_2 \frac{x^2}{1.2} + \ldots\ldots}{b_0 + b_1 \frac{x}{1} + b_2 \frac{x^2}{1.2} + \ldots\ldots} = A_0 + A_1 \frac{x}{1} + A_2 \frac{x^2}{1.2} + \ldots\ldots$$

$$A_0 = \frac{a_0}{b_0},$$

$$A_1 = \frac{b_0\, da_0 - a_0\, db_0}{b_0^2} = \frac{b_0\, a_1 - a_0\, b_1}{b_0^2},$$

$$A_2 = \frac{b_0 \{b_0\, a_2 - a_0\, b_2\} - 2b_1 \{b_0\, a_1 - a_0\, b_1\}}{b_0^3},$$

. . . .

As a particular example of the last theorem, suppose

$$u = f\left(y,\ \frac{dy}{dx},\ \frac{d^2y}{dx^2}, \ldots\ldots\right),$$

where $$y = a_0 + a_1 x + a_2 \frac{x^2}{1.2} + \ldots\ldots$$

$$\frac{dy}{dx} = a_1 + a_2 \frac{x}{1} + \ldots\ldots$$

$$\frac{d^2y}{dx^2} = a_2 + a_3 \frac{x}{1} + \ldots\ldots$$

. . . .

$$\therefore\ u = A_0 + A_1 x + A_2 \frac{x^2}{1.2} + \ldots\ldots$$

where $A_0 = f(a_0, a_1, a_2, \ldots\ldots)$

and A_1, A_2 ...... are the successive total differentials of A_0 on the supposition that $da_0 = a_1$, $da_1 = a_2$ ....

Thus if $$u = \frac{\frac{dy}{dx}}{y}, \qquad A_0 = \frac{a_1}{a_0},$$

$$A_1 = \frac{a_0\, da_1 - a_1\, da_0}{a_0^2} = \frac{a_0\, a_2 - a_1^2}{a_0^2},$$

. . . .

$$u = \frac{a_1}{a_0} + \frac{a_0\, a_2 - a_1^2}{a_0^2}\, \frac{x}{1} + \ldots\ldots$$

And the Theorem is capable of even greater extension. Thus let $u = f(v)$, where

$$v = a_{00} + a_{10}x + a_{01}y + \frac{1}{1.2}\left\{a_{20}x^2 + 2a_{11}xy + a_{02}y^2\right\} + \ldots (147)$$

then the coefficient of $\frac{1.2.3\ldots\ldots(m+n)}{1.2\ldots m\,1.2.3\ldots n}\,x^m\,y^n$ in u is found by differentiating $f(a_{00})$ $(m+n)$ times, that is, m times on the supposition that $d.a_{ij} = a_{i+1,j}$, and n times on the supposition that $d.a_{ij} = a_{i,j+1}$.

The preceding method of derivation is an extension of Taylor's Theorem; for whereas by the latter we are able to expand $\text{F}(a+x)$, by this we can develope, in ascending powers of x, $\text{F}(a_0 + a_1 x + a_2 \frac{x^2}{1.2} + \ldots\ldots)$. Numerous other examples will be found in Arbogast's work; the processes of which however are somewhat different to those explained above. The student also may consult the Treatise on the Differential and Integral Calculus, by Augustus De Morgan, London, 1842, Art. 214—227; the large treatise on the Differential Calculus, by S. F. Lacroix, Paris, 1810; and papers in the Cambridge and Dublin Mathematical Journal, by Mr. De Morgan, vol. i. p. 238 and vol. vi. p. 35, and by Professor Donkin, vol. vi. p. 141.

Section 7.—*Elimination of Constants and Functions by means of Differentiation.*

85.] Since a constant quantity connected with a variable by the symbols of addition and subtraction disappears in differentiation, it follows that if an equation can be put under the form

$$y = f(x) + c, \qquad (148)$$

its derived will be

$$\frac{dy}{dx} = f'(x);$$

from which the constant will have disappeared. And therefore, whatever may have been its value in the first or *primitive* equation, the derived equation contains no trace of it. Similarly, if an equation can be put under the form

$$y = f(x) + c_1 x + c_0,$$

$$\frac{dy}{dx} = f'(x) + c_1,$$

$$\frac{d^2y}{dx^2} = f''(x);$$

and thus in two differentiations two arbitrary constants will have disappeared. And similarly, if

$$y = f(x) + c_n x^n + c_{n-1} x^{n-1} + \dots\dots + c_2 x^2 + c_1 x + c_0,$$

$$\frac{dy}{dx} = f'(x) + nc_n x^{n-1} + \dots\dots + 2c_2 x + c_1,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\frac{d^{n+1} y}{dx^{n+1}} = f^{n+1}(x),$$

and thus all the $n+1$ arbitrary constants have disappeared in the process of $n+1$ differentiations; whenever therefore the functions admit of being put under the above forms, at each differentiation one arbitrary constant will disappear.

Ex. 1.
$$y = ax + b,$$

$$\frac{dy}{dx} = a,$$

$$\frac{d^2 y}{dx^2} = 0;$$

or substituting $\frac{dy}{dx}$ for a in the primitive,

$$y = x\frac{dy}{dx} + b,$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dx} + x\frac{d^2 y}{dx^2},$$

$$\therefore \quad \frac{d^2 y}{dx^2} = 0,$$

Whence it appears, that by one differentiation either a or b may be eliminated; and by two differentiations both may be made to disappear.

86.] Suppose however that the equation of relation between x and y is an implicit one, and involving m arbitrary constants, and of the form

$$u = \text{F}(x, y) = 0; \tag{149}$$

for the sake of simplicity consider x to be equicrescent, and differentiate (149) n times in succession; thereby n different equations will be formed, which, added to (149), give us $n+1$ different equations involving m constants. By means of these,

n of the constants may be eliminated, and (theoretically at least) any n of them; whereby the final equation will contain $m-n$ of the constants; and of course there may be as many different final equations as there are combinations of the m constants taken n and n together; that is, there may be

$$\frac{m(m-1)(m-2)\ldots\ldots(m-n+1)}{1.2.3\ldots\ldots n} \text{ different equations.}$$

Thus suppose (149) to be differentiated twice; three equations will then be given containing m constants. By means of which two may be eliminated, and the resulting equation will contain $m-2$ constants; and as any two may be eliminated, we may have as many different final equations as there are combinations of the m constants taken 2 and 2 together. Hence it follows, that if we differentiate m times, there will be altogether $m+1$ equations, from which the m arbitrary constants may be entirely eliminated; and that, *generally* m constants involved in the primitive cannot be eliminated, unless there be formed m derived-functions of the primitive expression.

The formation of equations involving derived-functions is of vast importance in a difficult branch of a future part of our work, and therefore we are giving to the origin of such equations closer consideration and more ample illustration.

Ex. 1. $y^2 = m(a^2-x^2)$; it is required to eliminate m and a.

$$y\frac{dy}{dx} = -mx; \tag{150}$$

also eliminating m between this and the primitive, we have

$$yx + (a^2-x^2)\frac{dy}{dx} = 0; \tag{151}$$

and differentiating again (150),

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = -m;$$

$$\therefore \quad xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0. \tag{152}$$

That is, we have two differential equations of the first order, (that is, containing $\frac{dy}{dx}$ only) which respectively do not involve a and m, and one differential equation of the second order, viz. (152), from which both the constants have disappeared.

Ex. 2. Eliminate the constant a from

$$(1-x^2)^{\frac{1}{2}} + (1-y^2)^{\frac{1}{2}} = a(x-y),$$

$$\frac{-x\,dx}{(1-x^2)^{\frac{1}{2}}} - \frac{y\,dy}{(1-y^2)^{\frac{1}{2}}} = a(dx-dy);$$

whence, by the common process of elimination, and dividing out the common factor

$$\frac{dx}{(1-x^2)^{\frac{1}{2}}} = \frac{dy}{(1-y^2)^{\frac{1}{2}}}.$$

Ex. 3. Eliminate c from

$$x^2 + y^2 = cx,$$

$$2(x\,dx + y\,dy) = c\,dx;$$

$$\therefore \quad c = 2\left(y\frac{dy}{dx} + x\right),$$

$$\therefore \quad x^2+y^2 = 2xy\frac{dy}{dx} + 2x^2,$$

$$y^2 - 2xy\frac{dy}{dx} - x^2 = 0.$$

Ex. 4. Given $(x-a)^2 + (y-\beta)^2 = \rho^2$; it is required to eliminate a and β by differentiation.

Let neither x nor y be equicrescent; then

$$(x-a)\,dx + (y-\beta)\,dy = 0,$$

$$(x-a)\,d^2x + (y-\beta)\,d^2y + dx^2 + dy^2 = 0;$$

$$\therefore \quad x-a = \frac{(dx^2+dy^2)\,dy}{dx\,d^2y - dy\,d^2x},$$

$$y-\beta = -\frac{(dx^2+dy^2)\,dx}{dx\,d^2y - dy\,d^2x};$$

$\therefore$ squaring and adding

$$(x-a)^2 + (y-\beta)^2 = \rho^2 = \frac{(dx^2+dy^2)^3}{(dx\,d^2y - dy\,d^2x)^2}.$$

And if x be equicrescent,

$$\rho^2 = \frac{(dx^2+dy^2)^3}{(dx\,d^2y)^2} = \frac{\left(1+\frac{dy^2}{dx^2}\right)^3}{\left(\frac{d^2y}{dx^2}\right)^2}.$$

87.] Also since the differentials of logarithmic and inverse-circular functions are algebraical quantities, of exponential functions reproduce themselves, and of circular functions are other circular quantities related to them by trigonometrical formulæ, such transcendental functions may be eliminated from given primitive equations by means of differentiation: the following examples are cited to explain the process, and enable the student to apply it to other similar problems.

Ex. 1. If $y = (a^2+x^2)^{\frac{m}{n}}$; it is required to eliminate the irrational function.

$$y = (a^2 + x^2)^{\frac{m}{n}},$$

$$\therefore \quad \log y = \frac{m}{n} \log (a^2 + x^2),$$

$$\frac{dy}{y} = \frac{m}{n} \frac{2x\, dx}{a^2 + x^2};$$

$$\therefore \quad \frac{dy}{dx} = \frac{2m\, xy}{n(a^2 + x^2)}:$$

an expression which is free from radical quantities.

Ex. 2.
$$y = a \sin x + b \cos x,$$

$$\frac{dy}{dx} = a \cos x - b \sin x,$$

$$\frac{d^2y}{dx^2} = - a \sin x - b \cos x;$$

$$\therefore \quad \frac{d^2y}{dx^2} + y = 0.$$

Ex. 3.
$$y = e^{\sin^{-1} x},$$

$$\frac{dy}{dx} = e^{\sin^{-1} x} \frac{1}{(1-x^2)^{\frac{1}{2}}},$$

$$= \frac{y}{(1-x^2)^{\frac{1}{2}}},$$

$$\therefore \quad (1-x^2)^{\frac{1}{2}} \frac{dy}{dx} = y;$$

$$\therefore \quad (1-x^2)^{\frac{1}{2}} \frac{d^2y}{dx^2} - \frac{x}{(1-x^2)^{\frac{1}{2}}} \frac{dy}{dx} = \frac{dy}{dx};$$

$$\therefore \quad (1-x^2) \frac{d^2y}{dx^2} - x\frac{dy}{dx} = (1-x^2)^{\frac{1}{2}} \frac{dy}{dx},$$

$$= y;$$

an expression which is free from radicals and transcendental functions.

Ex. 4. $$y = be^{ax} \cos(nx+c),$$

$$\frac{dy}{dx} = abe^{ax} \cos(nx+c) - nbe^{ax} \sin(nx+c),$$

$$= ay - nbe^{ax} \sin(nx+c),$$

$$\frac{d^2y}{dx^2} = a\frac{dy}{dx} - nbae^{ax} \sin(nx+c) - n^2be^{ax} \cos(nx+c),$$

$$= \ldots\ldots + a\left(\frac{dy}{dx} - ay\right) - n^2y;$$

$$\therefore \quad \frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2+n^2)y = 0;$$

an expression free from exponential and circular functions.

Ex. 5. $$y = \frac{e^x + e^{-x}}{e^x - e^{-x}};$$

$$\therefore \quad y = \frac{e^{2x}+1}{e^{2x}-1},$$

$$e^{2x} = \frac{y+1}{y-1}; \quad \therefore \quad 2x = \log(y+1) - \log(y-1),$$

$$2dx = \frac{dy}{y+1} - \frac{dy}{y-1};$$

$$\therefore \quad dx = \frac{dy}{1-y^2}, \quad \text{or } \frac{dy}{dx} = 1-y^2.$$

88.] Elimination of arbitrary functions by differentiation.

First, let us consider the simple case of an explicit function of two independent variables of the form

$$u = \text{F}(x, y) \tag{153}$$

in which the form of the function F is undetermined; let the

partial derived-functions of u be calculated, with respect to the variations of x and y, whereby two equations will be obtained involving the same derived-function, viz. $\mathrm{F}'(x, y)$, which may be eliminated by means of them; and thereby a final equation will be obtained involving $\left(\frac{du}{dx}\right)$ and $\left(\frac{du}{dy}\right)$, but independent of the primitive function. If the primitive contained many arbitrary functions of x and y, successive partial derived-functions of them may be formed, and equations may be obtained, independent of the arbitrary functions, but in terms of the partial derived-functions.

Ex. 1. $$u = ax + by + cf(mx + ny),$$

$$\left(\frac{du}{dx}\right) = a + cmf'(mx + ny),$$

$$\left(\frac{du}{dy}\right) = b + cnf'(mx + ny),$$

$$n\left(\frac{du}{dx}\right) - m\left(\frac{du}{dy}\right) = an - bm.$$

Ex. 2. $$u = \mathrm{F}(x^2 + y^2),$$

$$\left(\frac{du}{dx}\right) = 2x\,\mathrm{F}'(x^2 + y^2),$$

$$\left(\frac{du}{dy}\right) = 2y\,\mathrm{F}'(x^2 + y^2),$$

$$\therefore\ y\left(\frac{du}{dx}\right) - x\left(\frac{du}{dy}\right) = 0.$$

Ex. 3. $$u = \mathrm{F}\left(\frac{y}{x}\right),$$

$$\left(\frac{du}{dx}\right) = -\frac{y}{x^2}\,\mathrm{F}'\left(\frac{y}{x}\right),$$

$$\left(\frac{du}{dy}\right) = \frac{1}{x}\,\mathrm{F}'\left(\frac{y}{x}\right),$$

$$\therefore\ x\left(\frac{du}{dx}\right) + y\left(\frac{du}{dy}\right) = 0.$$

Which result is a particular case of one of Euler's Theorems of Homogeneous Functions.

Ex. 4. $$u = y^2 + 2f\left(\frac{1}{x} + \log y\right),$$

$$\left(\frac{du}{dx}\right) = -\frac{2}{x^2}f'\left(\frac{1}{x} + \log y\right),$$

$$\left(\frac{du}{dy}\right) = 2y + \frac{2}{y}f'\left(\frac{1}{x} + \log y\right);$$

$$\therefore \quad x^2\left(\frac{du}{dx}\right) + y\left(\frac{du}{dy}\right) = 2y^2.$$

Ex. 5. $$u = f(ax + by) + \phi(bx - ay,)$$

$$\left(\frac{du}{dx}\right) = af'(ax + by) + b\phi'(bx - ay),$$

$$\left(\frac{du}{dy}\right) = bf'(ax + by) - a\phi'(bx - ay);$$

$$\therefore \quad a\left(\frac{du}{dx}\right) + b\left(\frac{du}{dy}\right) = (a^2 + b^2)f'(ax + by).$$

And differentiating again:

$$a\left(\frac{d^2u}{dx^2}\right) + b\left(\frac{d^2u}{dx\,dy}\right) = a(a^2 + b^2)f''(ax + by),$$

$$a\left(\frac{d^2u}{dy\,dx}\right) + b\left(\frac{d^2u}{dy^2}\right) = b(a^2 + b^2)f''(ax + by);$$

$$\therefore \quad ab\left\{\left(\frac{d^2u}{dx^2}\right) - \left(\frac{d^2u}{dy^2}\right)\right\} + (b^2 - a^2)\left(\frac{d^2u}{dx\,dy}\right) = 0.$$

Ex. 6. $$u = xf\left(\frac{y}{x}\right) + F(xy),$$

$$\left(\frac{du}{dx}\right) = f\left(\frac{y}{x}\right) - \frac{y}{x}f'\left(\frac{y}{x}\right) + y\,F'(xy),$$

$$\left(\frac{du}{dy}\right) = f'\left(\frac{y}{x}\right) + x\,F'(xy);$$

$$\therefore \quad x\left(\frac{du}{dx}\right) + y\left(\frac{du}{dy}\right) = xf\left(\frac{y}{x}\right) + 2xy\,F'(xy);$$

and substituting for $f\left(\frac{y}{x}\right)$ from the given primitive, we have

$$x\left(\frac{du}{dx}\right)+y\left(\frac{du}{dy}\right)=u-\mathrm{F}(xy)+2xy\,\mathrm{F}'(xy),$$

$$=u-\phi(xy),$$

replacing $\mathrm{F}(xy)-2xy\,\mathrm{F}'(xy)$ by $\phi(xy)$;

$$\therefore\quad x\left(\frac{d^2u}{dx^2}\right)+\left(\frac{du}{dx}\right)+y\left(\frac{d^2u}{dx\,dy}\right)=\left(\frac{du}{dx}\right)-y\phi'(xy),$$

$$x\left(\frac{d^2u}{dx\,dy}\right)+\left(\frac{du}{dy}\right)+y\left(\frac{d^2u}{dy^2}\right)=\left(\frac{du}{dy}\right)-x\phi'(xy),$$

$$x^2\left(\frac{d^2u}{dx^2}\right)-y^2\left(\frac{d^2u}{dy^2}\right)=0.$$

89.] If, in the equation containing the arbitrary function, three variables, x, y, z, are *implicitly* involved, the same method as above may be followed in forming the partial derived-functions, if we consider one of the variables to be a function of the other two, and on this supposition calculate its partial variations due to the variations of the others.

Ex. 1. $\qquad lx-nz=f(my-nz).$

Consider z to be a function of two independent variables, x and y, and calculate $\left(\frac{dz}{dx}\right)$ and $\left(\frac{dz}{dy}\right)$:

$$l-n\left(\frac{dz}{dx}\right)=f'(my-nz)\left\{-n\left(\frac{dz}{dx}\right)\right\},$$

$$-n\left(\frac{dz}{dy}\right)=f'(my-nz)\left\{m-n\left(\frac{dz}{dy}\right)\right\},$$

$$\therefore\quad \frac{1}{l}\left(\frac{dz}{dx}\right)+\frac{1}{m}\left(\frac{dz}{dy}\right)=\frac{1}{n}.$$

This result also admits of being put into another form: suppose that the primitive expression, $lx-nz-f(my-nz)=0$, is written in the form

$$\mathrm{F}(x,y,z)=0,$$

then, by reason of the latter part of Art. 50, we may replace as follows:

$$\left(\frac{dz}{dx}\right) \text{ by } -\frac{\left(\frac{d\text{F}}{dx}\right)}{\left(\frac{d\text{F}}{dz}\right)},$$

$$\left(\frac{dz}{dy}\right) \text{ by } -\frac{\left(\frac{d\text{F}}{dy}\right)}{\left(\frac{d\text{F}}{dz}\right)};$$

whereby the partial differential equation becomes

$$\frac{1}{l}\left(\frac{d\text{F}}{dx}\right) + \frac{1}{m}\left(\frac{d\text{F}}{dy}\right) + \frac{1}{n}\left(\frac{d\text{F}}{dz}\right) = 0;$$

and this last result, it is to be observed, is independent of any supposition as to one of the three variables involved in the original primitive being dependent on the other two.

Ex. 2. $$\frac{y-b}{z-c} = f\left(\frac{x-a}{z-c}\right).$$

Let us consider z to be a function of x and y, and calculate the partial derived-functions of z on this supposition; then

$$-\frac{y-b}{(z-c)^2}\left(\frac{dz}{dx}\right) = f'\left(\frac{x-a}{z-c}\right)\frac{(z-c)-(x-a)\left(\frac{dz}{dx}\right)}{(z-c)^2},$$

$$\frac{(z-c)-(y-b)\left(\frac{dz}{dy}\right)}{(z-c)^2} = -f'\left(\frac{x-a}{z-c}\right)\frac{x-a}{(z-c)^2}\left(\frac{dz}{dy}\right);$$

whence, by division and reduction,

$$(x-a)\left(\frac{dz}{dx}\right) + (y-b)\left(\frac{dz}{dy}\right) = z-c.$$

By a similar process, if we had considered x to be a function of y and z, and had calculated the partial derived-functions $\left(\frac{dx}{dy}\right)$ and $\left(\frac{dx}{dz}\right)$, we should have found

$$x-a = \left(\frac{dx}{dz}\right)(z-c) + \left(\frac{dx}{dy}\right)(y-b).$$

Similarly also might y have been considered a function of x and z; and if the successive derived-functions had been formed on this supposition, we should have obtained a final equation of the same form as the two above.

Also if the primitive, viz.

$$\frac{y-b}{z-c} - f\left(\frac{x-a}{z-c}\right) = 0,$$

be considered in the form $F(x, y, z) = 0$,

and if in any one of the results substitutions analogous to those of the last example be made for, say, $\left(\frac{dz}{dx}\right)$ and $\left(\frac{dz}{dy}\right)$, the result is,

$$(x-a)\left(\frac{dF}{dx}\right) + (y-b)\left(\frac{dF}{dy}\right) + (z-c)\left(\frac{dF}{dz}\right) = 0.$$

Ex. 3. Eliminate the function from

$$x^2 + y^2 + z^2 = f(lx + my + nz).$$

Let z be a variable dependent on x and y, which are two independent variables,

$$2x + 2z\left(\frac{dz}{dx}\right) = \left\{l + n\left(\frac{dz}{dx}\right)\right\} f'(lx + my + nz),$$

$$2y + 2z\left(\frac{dz}{dy}\right) = \left\{m + n\left(\frac{dz}{dy}\right)\right\} f'(lx + my + nz),$$

$$\therefore \quad (mz-ny)\left(\frac{dz}{dx}\right) + (nx-lz)\left(\frac{dz}{dy}\right) = ly - mx.$$

And if the primitive be in the form $F(x, y, z) = 0$, the resulting partial differential equation is

$$(mz-ny)\left(\frac{dF}{dx}\right) + (nx-lz)\left(\frac{dF}{dy}\right) + (ly-mx)\left(\frac{dF}{dz}\right) = 0.$$

90.] The same principles may be extended to the elimination of indeterminate functions of any number of independent variables, and also to those of many variables which are related to each other by one or more equations of condition.

Ex. 1. $$u = xf\left(\frac{y}{x}, \frac{x}{z}\right),$$

$$\left(\frac{du}{dx}\right) = f\left(\frac{y}{x}, \frac{x}{z}\right) - \frac{y}{x}f'\left(\frac{y}{x}, \frac{x}{z}\right) + \frac{x}{z}f'\left(\frac{y}{x}, \frac{x}{z}\right),$$

$$\left(\frac{du}{dy}\right) = f'\left(\frac{y}{x}, \frac{x}{z}\right),$$

$$\left(\frac{du}{dz}\right) = -\frac{x^2}{z^2}f'\left(\frac{y}{x}, \frac{x}{z}\right);$$

$$\therefore\quad x\left(\frac{du}{dx}\right) + y\left(\frac{du}{dy}\right) + z\left(\frac{du}{dz}\right) = xf\left(\frac{y}{x}, \frac{x}{z}\right),$$
$$= u.$$

Ex. 2. $$u = f(ax^2+by^3+cz^4)+\phi(\cos lx+\cos my+\cos nz),$$

$$\left(\frac{du}{dx}\right) = 2axf'(ax^2+by^3+cz^4) - l\sin lx\phi'(\cos lx+\cos my+\cos nz),$$

$$\left(\frac{du}{dy}\right) = 3by^2f'(ax^2+by^3+cz^4) - m\sin my\phi'(\cos ly+\cos my+\cos nz),$$

$$\left(\frac{du}{dz}\right) = 4cz^3f'(ax^2+by^3+cz^4) - n\sin nz\phi'(\cos lx+\cos my+\cos nz);$$

$\therefore$ by Lagrange's method of cross-multiplication (see Preliminary Proposition II.) we have

$$\left(\frac{du}{dx}\right)\{4cmz^3\sin my-3bny^2\sin nz\}+\left(\frac{du}{dy}\right)\{2anx\sin nz-4clz^3\sin lx\} + \left(\frac{du}{dz}\right)\{3bly^2\sin lx-2amx\sin my\} = 0,$$

an equation independent of the arbitrary functions, and therefore expressive of the properties of such functions, whatever be their specific form. By similar methods we may eliminate arbitrary functions of any number of variables.

91.] In general, for determining to what order of differentiation we must proceed to eliminate any number of arbitrary

functions from an expression containing two variables, let the following considerations suffice.

Suppose $u = 0$ to comprise m arbitrary functions of x and y, then it is plain that each successive differentiation introduces m other arbitrary functions, which are the derived of the given functions; so that by proceeding to the nth order of differentiation, we have $(n+1)m$ different arbitrary functions: but as the original equation $u = 0$ gives one relation amongst these functions, so do

$$\left(\frac{du}{dx}\right) = 0, \qquad \left(\frac{du}{dy}\right) = 0,$$

$$\left(\frac{d^2u}{dx^2}\right) = 0, \qquad \left(\frac{d^2u}{dxdy}\right) = 0, \qquad \left(\frac{d^2u}{dy^2}\right) = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\left(\frac{d^nu}{dx^n}\right) = 0, \qquad \left(\frac{d^nu}{dx^{n-1}dy}\right) = 0, \ldots\ldots \left(\frac{d^nu}{dy^n}\right) = 0,$$

give us other relations; and therefore by means of n differentiations we have the number of relations equal to

$$1+2+3+\ .\ .\ .\ .\ .\ .\ +(n+1) = \frac{(n+1)(n+2)}{2}.$$

And in order that we may be able to eliminate all these, we must evidently have the number of relations greater than the number of unknown quantities, that is

$$\frac{(n+1)(n+2)}{1.2} > (n+1)m;$$

that is $\qquad n+2 > 2m,$

$$n > 2m-2;$$

that is, n, which expresses the order of differentiation, must $= 2m-1$ at least; and we shall then have a sufficient number of equations to eliminate the arbitrary functions from. Thus, if the original equation involve but one arbitrary function, $m = 1$, and we need differentiate but once; if it involve two arbitrary functions, we must in the general case differentiate thrice, and so on. An example is subjoined in which three differentiations are required:

$$u = f(x+y) + xy\phi(x-y).$$

$$\left(\frac{du}{dx}\right) = f'(x+y) + y\phi(x-y) + xy\phi'(x-y),$$

$$\left(\frac{du}{dy}\right) = f'(x+y) + x\phi(x-y) - xy\phi'(x-y),$$

$$\left(\frac{du}{dx}\right) - \left(\frac{du}{dy}\right) = (y-x)\,\phi(x-y) + 2\,xy\phi'(x-y),$$

$$\therefore\ \left(\frac{d^2u}{dx^2}\right) - \left(\frac{d^2u}{dx\,dy}\right) = (y-x)\,\phi'(x-y) - \phi(x-y) + 2\,y\phi'(x-y) + 2\,xy\phi''(x-y),$$

$$\left(\frac{d^2u}{dx\,dy}\right) - \left(\frac{d^2u}{dy^2}\right) = -(y-x)\,\phi'(x-y) + \phi(x-y) + 2\,x\phi'(x-y) - 2\,xy\phi''(x-y);$$

$$\therefore\ \left(\frac{d^2u}{dx^2}\right) - \left(\frac{d^2u}{dy^2}\right) = 2(x+y)\phi'(x-y).$$

And differentiating again,

$$\left(\frac{d^3u}{dx^3}\right) - \left(\frac{d^3u}{dx\,dy^2}\right) = 2\phi'(x-y) + 2(x+y)\phi''(x-y),$$

$$\left(\frac{d^3u}{dy\,dx^2}\right) - \left(\frac{d^3u}{dy^3}\right) = 2\phi'(x-y) - 2(x+y)\phi''(x-y),$$

$$\therefore\ \left(\frac{d^3u}{dx^3}\right) + \left(\frac{d^3u}{dx^2\,dy}\right) - \left(\frac{d^3u}{dx\,dy^2}\right) - \left(\frac{d^3u}{dy^3}\right) = 4\phi'(x-y);$$

$$\text{but}\quad \phi'(x-y) = \frac{1}{2(x+y)}\left\{\left(\frac{d^2u}{dx^2}\right) - \left(\frac{d^2u}{dy^2}\right)\right\},$$

$$\therefore\ \left(\frac{d^3u}{dx^3}\right) + \left(\frac{d^3u}{dx^2\,dy}\right) - \left(\frac{d^3u}{dx\,dy^2}\right) - \left(\frac{d^3u}{dy^3}\right) = \frac{2}{x+y}\left\{\left(\frac{d^2u}{dx^2}\right) - \left(\frac{d^2u}{dy^2}\right)\right\}.$$

92.] In the latter Articles of this Chapter methods have been considered for deriving partial derived-functions from functions of two or more independent variables, and some of the uses of such derived-functions have been explained. Suppose however that by means of given equations the variables which entered into the function are expressed in terms of

other new variables, it is then necessary to inquire how substitutions of these are to be effected in the partial derived-functions; and as one or other of the original variables have been considered equicrescent, the problem becomes more complicated in making the requisite substitutions; the principles however of the present Chapter are sufficient for the purpose.

Let us first consider the more simple case of a function of two variables, viz. $u = f(x, y)$; and suppose an equation involving $\left(\frac{du}{dx}\right)$, $\left(\frac{du}{dy}\right)$, dx, dy; $\left(\frac{du}{dx}\right)$ and $\left(\frac{du}{dy}\right)$ having been calculated on the supposition that x varied while y remained constant, and y varied while x was constant, and dx and dy being partial changes in x and y on similar suppositions. It is manifest that the very fact of there being such expressions as $\left(\frac{du}{dx}\right)$ and $\left(\frac{du}{dy}\right)$, imports that there is such an expression as $u = f(x, y)$; whether the expression can be found or not is immaterial.

Suppose that the variables x and y are connected with other new variables, say r and θ, by means of equations of the form

$$x = \phi_1(r, \theta),$$
$$y = \phi_2(r, \theta);$$

if we substitute these values of x and y in the expressed or understood (as the case may be) function, the equation becomes of the form

$$u = \mathrm{F}(r, \theta);$$

the total differential of which is

$$\mathrm{D}u = \left(\frac{d\mathrm{F}}{dr}\right) dr + \left(\frac{d\mathrm{F}}{d\theta}\right) d\theta;$$

whence, dividing through successively by dx and dy, and changing $\frac{\mathrm{D}u}{dx}$ into $\left(\frac{du}{dx}\right)$, and $\frac{\mathrm{D}u}{dy}$ into $\left(\frac{du}{dy}\right)$, since in these cases we have the ratio of the partial variations of u and of x, and of u and of y, and bracketing them to indicate that they are partial, we have

$$\left(\frac{du}{dx}\right) = \left(\frac{d\mathrm{F}}{dr}\right)\frac{dr}{dx} + \left(\frac{d\mathrm{F}}{d\theta}\right)\frac{d\theta}{dx},$$
$$\left(\frac{du}{dy}\right) = \left(\frac{d\mathrm{F}}{dr}\right)\frac{dr}{dy} + \left(\frac{d\mathrm{F}}{d\theta}\right)\frac{d\theta}{dy}$$

remembering that $\frac{dr}{dx}$, $\frac{d\theta}{dx}$ are to be calculated on the supposition that y does not vary, that is, that $dy = 0$; and $\frac{dr}{dy}$, $\frac{d\theta}{dy}$, on the supposition that $dx = 0$. Since then, from the two equations given above, we have

$$dx = \left(\frac{d\phi_1}{dr}\right) dr + \left(\frac{d\phi_1}{d\theta}\right) d\theta,$$

$$dy = \left(\frac{d\phi_2}{dr}\right) dr + \left(\frac{d\phi_2}{d\theta}\right) d\theta;$$

therefore to calculate dx, let $dy = 0$, and eliminating $d\theta$ and dr between these two equations on this supposition, we have

$$\frac{dx}{dr} = \frac{\frac{d\phi_1}{dr}\frac{d\phi_2}{d\theta} - \frac{d\phi_2}{dr}\frac{d\phi_1}{d\theta}}{\frac{d\phi_2}{d\theta}},$$

$$\frac{dx}{d\theta} = \frac{\frac{d\phi_1}{d\theta}\frac{d\phi_2}{dr} - \frac{d\phi_2}{d\theta}\frac{d\phi_1}{dr}}{\frac{d\phi_2}{dr}};$$

similarly,
$$\frac{dy}{dr} = \frac{\frac{d\phi_1}{d\theta}\frac{d\phi_2}{dr} - \frac{d\phi_2}{d\theta}\frac{d\phi_1}{dr}}{\frac{d\phi_1}{d\theta}},$$

$$\frac{dy}{d\theta} = \frac{\frac{d\phi_2}{d\theta}\frac{d\phi_1}{dr} - \frac{d\phi_1}{d\theta}\frac{d\phi_2}{dr}}{\frac{d\phi_1}{dr}};$$

all these differential coefficients being partial; and if we substitute these quantities in the expressions above for $\left(\frac{du}{dx}\right)$ and $\left(\frac{du}{dy}\right)$, the resulting expressions will be the equivalents of $\left(\frac{du}{dx}\right)$ and $\left(\frac{du}{dy}\right)$, when x and y are replaced by their equivalents in terms of r and θ.

93.] Ex. 1. To transform $\left(\frac{d\text{R}}{dx}\right)$ and $\left(\frac{d\text{R}}{dy}\right)$ into their equivalents, having given

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{154}$$

In this problem there is implied a function, $\text{R} = f(x, y)$, which becomes, when x and y are replaced by their equivalents,

$$\text{R} = \text{F}(r, \theta);$$

$$\left.\begin{aligned} \therefore \quad \left(\frac{d\text{R}}{dx}\right) &= \left(\frac{d\text{R}}{dr}\right)\frac{dr}{dx} + \left(\frac{d\text{R}}{d\theta}\right)\frac{d\theta}{dx}, \\ \left(\frac{d\text{R}}{dy}\right) &= \left(\frac{d\text{R}}{dr}\right)\frac{dr}{dy} + \left(\frac{d\text{R}}{d\theta}\right)\frac{d\theta}{dy}. \end{aligned}\right\} \tag{155}$$

$$\left.\begin{aligned} dx &= dr\cos\theta - r\sin\theta d\theta, \\ dy &= dr\sin\theta + r\cos\theta d\theta. \end{aligned}\right\} \tag{156}$$

To calculate dx, dy must be equal to 0; whence, eliminating $d\theta$ and dr in turn from the equations

$$dx = dr\cos\theta - r\sin\theta d\theta,$$
$$0 = dr\sin\theta + r\cos\theta d\theta,$$

we have $$\frac{dx}{dr} = \frac{1}{\cos\theta}, \qquad \frac{dx}{d\theta} = -\frac{r}{\sin\theta}. \tag{157}$$

Similarly to calculate dy, $dx = 0$; wherefore, by means of

$$0 = dr\cos\theta - r\sin\theta d\theta,$$
$$dy = dr\sin\theta + r\cos\theta d\theta,$$

we have $$\frac{dy}{dr} = \frac{1}{\sin\theta}, \qquad \frac{dy}{d\theta} = \frac{r}{\cos\theta}; \tag{158}$$

and substituting these values in (155),

$$\left.\begin{aligned} \left(\frac{d\text{R}}{dx}\right) &= \left(\frac{d\text{R}}{dr}\right)\cos\theta - \left(\frac{d\text{R}}{d\theta}\right)\frac{\sin\theta}{r}, \\ \left(\frac{d\text{R}}{dy}\right) &= \left(\frac{d\text{R}}{dr}\right)\sin\theta + \left(\frac{d\text{R}}{d\theta}\right)\frac{\cos\theta}{r}; \end{aligned}\right\} \tag{159}$$

whence we have two transformations useful in the Planetary Theory, viz.

$$x\left(\frac{d\text{R}}{dy}\right) - y\left(\frac{d\text{R}}{dx}\right) = \left(\frac{d\text{R}}{d\theta}\right),$$

$$x\left(\frac{d\text{R}}{dx}\right) + y\left(\frac{d\text{R}}{dy}\right) = r\left(\frac{d\text{R}}{dr}\right).$$

Ex. 2. To transform $\left(\frac{d^2 v}{dx^2}\right)+\left(\frac{d^2 v}{dy^2}\right)$ into its equivalent in terms of r and θ, having given $x = r\cos\theta$, $y = r\sin\theta$.

By the same process as that employed in the last example, we have

$$\left(\frac{dv}{dx}\right)=\left(\frac{dv}{dr}\right)\cos\theta-\left(\frac{dv}{d\theta}\right)\frac{\sin\theta}{r}, \tag{160}$$

$$\left(\frac{dv}{dy}\right)=\left(\frac{dv}{dr}\right)\sin\theta+\left(\frac{dv}{d\theta}\right)\frac{\cos\theta}{r}; \tag{161}$$

and differentiating (160), bearing in mind that $\left(\frac{dv}{dr}\right)$ and $\left(\frac{dv}{d\theta}\right)$ are functions of r and θ, we have

$$\begin{aligned}\left(\frac{d^2 v}{dx^2}\right) &= \cos\theta\left\{\left(\frac{d^2 v}{dr^2}\right)\frac{dr}{dx}+\left(\frac{d^2 v}{dr\,d\theta}\right)\frac{d\theta}{dx}\right\}\\ &\quad-\frac{\sin\theta}{r}\left\{\left(\frac{d^2 v}{dr\,d\theta}\right)\frac{dr}{dx}+\left(\frac{d^2 v}{d\theta^2}\right)\frac{d\theta}{dx}\right\}\\ &\quad-\left(\frac{dv}{dr}\right)\sin\theta\frac{d\theta}{dx}-\left(\frac{dv}{d\theta}\right)\frac{r\cos\theta\,d\theta-\sin\theta\,dr}{r^2\,dx};\end{aligned}$$

whence, substituting from (157) and (158), we have

$$\begin{aligned}\left(\frac{d^2 v}{dx^2}\right) &= (\cos\theta)^2\left(\frac{d^2 v}{dr^2}\right)-\frac{2\sin\theta\cos\theta}{r}\left(\frac{d^2 v}{dr\,d\theta}\right)+\frac{(\sin\theta)^2}{r^2}\left(\frac{d^2 v}{d\theta^2}\right)\\ &\quad+\left(\frac{dv}{dr}\right)\frac{(\sin\theta)^2}{r}+\frac{2\sin\theta\cos\theta}{r^2}\left(\frac{dv}{d\theta}\right).\end{aligned} \tag{162}$$

Similarly,

$$\begin{aligned}\left(\frac{d^2 v}{dy^2}\right) &= (\sin\theta)^2\left(\frac{d^2 v}{dr^2}\right)+\frac{2\sin\theta\cos\theta}{r}\left(\frac{d^2 v}{dr\,d\theta}\right)+\frac{(\cos\theta)^2}{r^2}\left(\frac{d^2 v}{d\theta^2}\right)\\ &\quad+\left(\frac{dv}{dr}\right)\frac{(\cos\theta)^2}{r}-\frac{2\sin\theta\cos\theta}{r^2}\left(\frac{dv}{d\theta}\right);\end{aligned} \tag{163}$$

$$\therefore\ \left(\frac{d^2 v}{dx^2}\right)+\left(\frac{d^2 v}{dy^2}\right)=\left(\frac{d^2 v}{dr^2}\right)+\frac{1}{r^2}\left(\frac{d^2 v}{d\theta^2}\right)+\frac{1}{r}\left(\frac{dv}{dr}\right). \tag{164}$$

The results (162) and (164) are no other than particular cases of equation (100), Art. 74, when the right substitutions are made, and consistently with the independence and equicrescence of the variables.

Ex. 3. Having given $x = r\cos\theta$, $y = r\sin\theta$, to transform $dx\,dy$ into its equivalent, subject to the conditions that y does not vary when x varies, and x does not vary when y varies.

$$dx = dr\cos\theta - r\sin\theta d\theta,$$

$$dy = dr\sin\theta + r\cos\theta d\theta;$$

$$\therefore \text{ as before, } \qquad dx = \frac{1}{\cos\theta}\,dr.$$

Whence it follows, that when $dx = 0$, that is, when y varies, $dr = 0$.

$$\therefore \quad dy = r\cos\theta d\theta;$$

$$\therefore \quad dy\,dx = r\,dr\,d\theta.$$

If the expression to be transformed involves three variables x, y, z, and these are given in terms of three other variables r, θ, ϕ, the operations are to be effected in a similar manner; but as a consideration of the particular forms of the functions connecting the variables will usually shorten the process, and as the principle is the same as that involved in the last Article, it is of little use to calculate the general expressions, and therefore we forbear to give them.

CHAPTER IV.

ON THE RELATIONS BETWEEN FUNCTIONS AND DERIVED-FUNCTIONS, ON WHICH CERTAIN APPLICATIONS OF THE CALCULUS DEPEND.

94.] In the preceding part of the work, with the single exception of Art. 66—68, we have considered the changes of the variables to be *infinitesimal*, and have estimated the changes of the functions and their derivatives as they are due to such infinitesimal increments; but many subsequent applications require a knowledge of the properties of functions, when the variables are increased by *finite* augments. One object of this Chapter is therefore to connect such finite changes with differentials and derived-functions; a relation of the kind has been established in equation (81) of Art. 67, the second member of which consists of a finite number of terms; and an accurate equality exists between the sum of them and the left-hand member, except so far as θ is an undetermined proper fraction. But the proof is insufficient, inasmuch as it does not afford any answers to such questions as follow: Are *all* functions of the form $f(x+h)$ capable of expansion in a series of the form of the second member of equation (81); and if all are not, what are the characteristics of those which are? and supposing the function to be capable of expansion, can it be so expanded for all values of x and h, or is it possible for some and impossible for others? and what are the notes of x and h for which it is possible? and is the series true when continued to any number of terms, or must it cease at a certain term, because the requisite conditions are not satisfied by subsequent terms? Hence arises the necessity of proving certain theorems which establish relations between functions and their derivatives, and which involve certain conditions subject to which they are, and in failure of which the functions are not, within the general range of the Calculus; and by means of these, the infinitesimal calculus will be extended to changes of functions due to the finite changes of the variables.

95.] THEOREM I.—Given that $y = f(x)$ is a continuous function of x, for a given value of it, viz. $x = x_0$, and for values a little greater and a little less than x_0, that is, for $x_0 + dx$, and $x_0 - dx$: then, if $f'(x_0)$ is positive, x and $f(x)$ are, for that particular value, increasing or decreasing simultaneously; and if $f'(x_0)$ is negative, as x increases and passes through x_0, $f(x)$ is decreasing, or *vice versâ*.

Let Δy and Δx be, as before, the simultaneous and finite changes in the values of y and x; then it is plain, that according as $\frac{\Delta y}{\Delta x}$ is positive or negative when the increments are less than any assignable quantity, so is $\frac{dy}{dx}$ or $f'(x)$ which is its limit.

Since,
$$y = f(x),$$
$$y + \Delta y = f(x + \Delta x),$$
$$\therefore \quad \frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{(x + \Delta x) - x}.$$

On the supposition that $\frac{\Delta y}{\Delta x}$ is positive, the numerator and denominator of the fraction must have the same signs; and therefore, if $x + \Delta x$ is $> x$, that is, if x increases, $f(x + \Delta x)$ is $>$ than $f(x)$: but if $x + \Delta x$ is $< x$, that is, if x decreases, then $f(x + \Delta x)$ is $<$ than $f(x)$; and the same is true of the limiting value when Δx and Δy become dx and dy. So again if $\frac{\Delta y}{\Delta x}$ be negative, the numerator and denominator must have different signs; and therefore, if x increases, $f(x)$ must decrease: and if x decreases, $f(x)$ increases, and the same will be true in the limit. Hence we conclude that, if $\frac{dy}{dx} = f'(x)$ be positive for $x = x_0$, at that particular value, x and $f(x)$ are increasing or decreasing simultaneously; and if $f'(x)$ is negative when $x = x_0$, as x increases $f(x)$ decreases, or as x decreases $f(x)$ increases.

COROLLARY I.—Hence if $f(x)$ be continuous for every value of x between x_0 and x_n (x_n being greater than x_0), x and $f(x)$ are increasing or decreasing simultaneously through all values

for which $f'(x)$ is positive; and through all values for which $f'(x)$ is negative, as x increases, $f(x)$ decreases, and *vice versâ*.

COR. II.—Hence also if, up to a certain value, $x = a$, as x increases $f(x)$ increases, and after that value $f(x)$ decreases, $f'(x)$ will be positive until $x = a$, and then will become negative; and as the sign of such a quantity changes only by the quantity passing through zero or infinity (according as the factor, to the change of sign of which the function's change of sign is due, is in the numerator or denominator), so, when $x = a$, will $f'(x)$ be equal to either zero or infinity.

In illustration of this Theorem and its Corollaries, let us consider the following examples:

In fig. 10. Let OBA be a semicircle, the radius of which is equal to a; let O be the origin and OCA the axis of x, then the equation to it is

$$y^2 = 2ax - x^2,$$

$$f(x) = y = \{2ax - x^2\}^{\frac{1}{2}},$$

$$\frac{dy}{dx} = f'(x) = \frac{a-x}{(2ax - x^2)^{\frac{1}{2}}}.$$

Now for all values of x between 0 and a, $f'(x)$ is positive, and therefore as x increases $f(x)$ increases also; and for all values of x between a and $2a$, $f'(x)$ is negative, and therefore as x increases, $f(x)$ decreases. And when $x = a, f'(x) = 0$, and changes sign from $+$ to $-$.

Or again, let $y = f(x) = \sin x$,

$$\therefore \quad \frac{dy}{dx} = f'(x) = \cos x.$$

Then for all values of x in the first quadrant, $\cos x$ is positive, and x and $\sin x$ are simultaneously increasing or decreasing. But for all values of x in the second and third quadrants, $f'(x)$ or $\cos x$ is negative, and $\sin x$ decreases as x increases; and similarly for all other values of the arc.

It is also to be observed, that not only by its sign does $\frac{dy}{dx}$ or $f'(x)$ indicate whether an increase of the variable is accompanied contemporaneously by an increase or decrease of the function, but it also by its value denotes the *rate* of such

increase or decrease; the greater $f'(x)$ is, if it be positive, the faster does $f(x)$ increase as x increases; and the less $f'(x)$ is, if it be negative, the slower does $f(x)$ decrease as x increases. Thus if $y = x$ (the equation to a straight line passing through the origin) $dy = dx$, and the simultaneous increments of x and y are equal; but if $y = 2x$, $dy = 2dx$, and the increment of y is twice that of x.

96.] THEOREM II.—If x_n and x_0 are two definite values of x, x_n being greater than x_0, and $x_n - x_0$ being a finite quantity, and if $F(x)$ be a function of x, which, as also its first-derived function, is finite and continuous for all values of x between x_n and x_0, then

$$F(x_n) - F(x_0) = (x_n - x_0)\, F'\{x_0 + \theta(x_n - x_0)\},$$

θ being some proper positive fraction.

Let the difference $x_n - x_0$ be divided into n parts, and let $x_1, x_2, \ldots\ldots x_{n-1}$ be the values of x corresponding to the $n-1$ points of division; and let us moreover suppose n to be so large, that each of the divided elements, $x_1 - x_0$, $x_2 - x_1, \ldots\ldots$ $x_n - x_{n-1}$, is an infinitesimal.

Then, observing the definition of a derived-function, we have the following series of equations:

$$\left.\begin{aligned} F(x_1) - F(x_0) &= (x_1 - x_0)\, F'(x_0), \\ F(x_2) - F(x_1) &= (x_2 - x_1)\, F'(x_1), \\ &\ldots\ldots\ldots\ldots \\ F(x_n) - F(x_{n-1}) &= (x_n - x_{n-1})\, F'(x_{n-1}); \end{aligned}\right\} \quad (1)$$

whence, adding all the first and second members of the series of equations, the sum of the first is $F(x_n) - F(x_0)$, and the sum of the second is, by Preliminary Theorem III, the product of the sum of the first factors, viz. $x_n - x_0$, and some mean value of the second factors, that is,

$$F(x_n) - F(x_0) = (x_n - x_0)\, F'\{x_0 + \theta(x_n - x_0)\}, \quad (2)$$

in which θ is some positive proper fraction: and therefore the last factor is a correct symbol of the required quantity; for putting $\theta = 0$, we have only the first term of the series, and which would therefore be right if all the derived-functions in equations (1) were equal; and if $\theta = 1$, we have $F'(x_n)$, which is

a term just beyond that which the series reaches; hence θ must be greater than zero and less than unity.

Let the finite difference $x_n - x_0$ be represented by h, so that

$$x_n - x_0 = h; \qquad \therefore \quad x_n = x_0 + h,$$

and h is the finite increment of x_0; then

$$\mathrm{F}(x_0 + h) - \mathrm{F}(x_0) = h\,\mathrm{F}'(x_0 + \theta h). \tag{3}$$

It is to be observed, that if h is infinitesimal, we must neglect θh when added to the finite quantity x_0; and the result is, the derived-function as originally described in Art. 18.

97.] THEOREM III.—If x_n and x_0 are two definite values of x, x_n being greater than x_0 and $x_n - x_0$ being a finite quantity; and if $\mathrm{F}(x)$ and $f(x)$ are functions of x, which, as also their first-derived-functions, are finite and continuous for all values of x between x_0 and x_n; and besides, if $f'(x)$ does not change sign within these limits, then

$$\frac{\mathrm{F}(x_n) - \mathrm{F}(x_0)}{f(x_n) - f(x_0)} = \frac{\mathrm{F}'\{x_0 + \theta\,(x_n - x_0)\}}{f'\{x_0 + \theta\,(x_n - x_0)\}},$$

where θ represents a fraction mean to 0 and 1.

Let the difference $x_n - x_0$ be divided into n parts, and x_1, x_2, ... x_{n-1} be the values of x corresponding to the $n-1$ points of division; and let us moreover suppose n to be infinite, so that each of the divided elements, $x_1 - x_0$, $x_2 - x_1$, ... $x_n - x_{n-1}$, may be an infinitesimal. Then, since

$$\mathrm{F}(x_1) - \mathrm{F}(x_0) = (x_1 - x_0)\,\mathrm{F}'(x_0),$$

and

$$f(x_1) - f(x_0) = (x_1 - x_0)\,f'(x_0);$$

$$\left.\begin{aligned}
\therefore \quad & \frac{\mathrm{F}(x_1) - \mathrm{F}(x_0)}{f(x_1) - f(x_0)} = \frac{\mathrm{F}'(x_0)}{f'(x_0)}. \\
\text{Similarly,} \quad & \frac{\mathrm{F}(x_2) - \mathrm{F}(x_1)}{f(x_2) - f(x_1)} = \frac{\mathrm{F}'(x_1)}{f'(x_1)}, \\
& \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
& \frac{\mathrm{F}(x_n) - \mathrm{F}(x_{n-1})}{f(x_n) - f(x_{n-1})} = \frac{\mathrm{F}'(x_{n-1})}{f'(x_{n-1})}.
\end{aligned}\right\} \tag{4}$$

Now this being a series of fractions whose denominators are all of the same sign, according to Preliminary Theorem IV, the ratio of the sum of all the numerators of the left-hand

members of the equations to the sum of all the denominators is equal to some mean value of the fractions; that is,

$$\frac{\mathrm{F}(x_n) - \mathrm{F}(x_0)}{f(x_n) - f(x_0)}$$

is equal to some mean value of the fractions which are the right-hand members of the above equations; and such a value will be properly represented by

$$\frac{\mathrm{F}'\{x_0 + \theta\,(x_n - x_0)\}}{f'\{x_0 + \theta\,(x_n - x_0)\}},$$

where θ is a proper and positive fraction; therefore

$$\frac{\mathrm{F}(x_n) - \mathrm{F}(x_0)}{f(x_n) - f(x_0)} = \frac{\mathrm{F}'\{x_0 + \theta\,(x_n - x_0)\}}{f'\{x_0 + \theta\,(x_n - x_0)\}}.$$

For the sake of convenience let $x_n - x_0 = h$, so that $x_n = x_0 + h$; and therefore h is the finite increment of x_0, and the functions under consideration include those for all values of x between x_0 and $x_0 + h$; then

$$\frac{\mathrm{F}(x_0 + h) - \mathrm{F}(x_0)}{f(x_0 + h) - f(x_0)} = \frac{\mathrm{F}'(x_0 + \theta h)}{f'(x_0 + \theta h)}. \tag{5}$$

The further condition to which $f(x)$ is subject, viz. that $f'(x)$ does not change sign between x_0 and $x_0 + h$, is necessary, in order that the sum of the denominators of the right-hand members of (4) may not vanish, for thereby the first member of (5) might be equal to an infinite quantity, and the equation might be useless.

To enable a student the better to grasp the full meaning of the conditions of this important theorem, the graphical illustration of fig. 11 is given.

Of the two curves therein delineated, let the upper one be that whose equation is $y = \mathrm{F}(x)$, and let the lower one represent $y = f(x)$.

Let O be the origin, $\mathrm{OM}_0 = x_0$, $\mathrm{OM}_n = x_n$, $\therefore\ \mathrm{M}_n\mathrm{M}_0 = h$; $\mathrm{OM} = x$, x referring to any point intermediate to M_0 and M_n; and suppose that $f'(x)$ is positive for all values of x between x_0 and x_n, which property is represented by the curve being such that the ordinate increases as the abscissa increases; now it is immaterial what forms, or branches, or points of discontinuity the curves may have outside the assigned limits; we consider them only within and between the limits, and restrict

them to certain conditions within them, and which are expressed in the curves of the figure, viz. that they are to be finite and continuous; also

$$\text{M}_0\text{P}_0 = \text{F}(x_0), \qquad \text{M}_0 p_0 = f(x_0),$$

$$\text{M}_n\text{P}_n = \text{F}(x_n) = \text{F}(x_0 + h), \qquad \text{M}_n p_n = f(x_n) = f(x_0 + h).$$

Cor. I.—Suppose in the above equation (5), $\text{F}(x_0) = 0$ and $f(x_0) = 0$, which expresses the condition that both the curves in fig. 11 pass through the point M_0, then

$$\frac{\text{F}(x_0 + h)}{f(x_0 + h)} = \frac{\text{F}'(x_0 + \theta h)}{f'(x_0 + \theta h)}. \tag{6}$$

Cor. II.—Also suppose that x_0, which is the lower limit of x, $= 0$, which expresses the condition that in fig. 11 the origin of coordinates is at M_0, then

$$\frac{\text{F}(h) - \text{F}(0)}{f(h) - f(0)} = \frac{\text{F}'(\theta h)}{f'(\theta h)},$$

and writing x for h, as h is measured from the origin M_0 in this case, we have

$$\frac{\text{F}(x) - \text{F}(0)}{f(x) - f(0)} = \frac{\text{F}'(\theta x)}{f'(\theta x)}; \tag{7}$$

and if $\text{F}(0) = 0$, and $f(0) = 0$, that is, if both the curves pass through the origin,

$$\frac{\text{F}(x)}{f(x)} = \frac{\text{F}'(\theta x)}{f'(\theta x)}. \tag{8}$$

98.] By Corollary I of the last Article and equation (6), it appears that if any two functions of x vanish when $x = x_0$, then, subject to the necessary conditions,

$$\frac{\text{F}(x_0 + h)}{f(x_0 + h)} = \frac{\text{F}'(x_0 + \theta h)}{f'(x_0 + \theta h)} = \frac{\text{F}'(x_0 + h_1)}{f'(x_0 + h_1)}; \tag{9}$$

replacing θh by h_1 for the sake of convenience, and observing therefore that h_1 is less than h.

Suppose now that $\text{F}'(x_0) = 0$, and $f'(x_0) = 0$, and that $\text{F}''(x)$ and $f''(x)$, as well as $\text{F}'(x)$ and $f'(x)$, are finite and continuous for all values of x between x_0 and $x_0 + \theta h$, and that $f''(x)$ does not change sign within these limits, then, by virtue of (9), we have

$$\frac{\text{F}'(x_0 + h_1)}{f'(x_0 + h_1)} = \frac{\text{F}''(x_0 + \theta h_1)}{f''(x_0 + \theta h_1)};$$

and replacing θh_1 by h_2, observing that h_2 is less than h_1, we have

$$\frac{\text{F}'(x_0 + h_1)}{f'(x_0 + h_1)} = \frac{\text{F}''(x_0 + h_2)}{f''(x_0 + h_2)}. \tag{10}$$

Similarly again, if $\text{F}''(x_0) = 0$ and $f''(x_0) = 0$, and if their derived-functions $\text{F}'''(x)$, $f'''(x)$ are finite and continuous for all values of x between x_0 and $x_0 + h_2$, and if, besides, $f''(x)$ always increases or decreases with x between these limits, then

$$\frac{\text{F}''(x_0 + h_2)}{f''(x_0 + h_2)} = \frac{\text{F}'''(x_0 + \theta h_2)}{f'''(x_0 + \theta h_2)} = \frac{\text{F}'''(x_0 + h_3)}{f'''(x_0 + h_3)}, \tag{11}$$

replacing θh_2 by h_3; whence it appears, that h_3, which is less than h_2, which is also less than h_1, and therefore less than h, is of the form θh.

Suppose now that these several conditions as to the derived-functions, and others similar to them, hold good up to the $(n-1)$th derived-functions inclusively, then we have the following proposition:

THEOREM IV.—If $\text{F}(x)$ and $f(x)$ are two functions of x, which, as also all the derived-functions up to the nth inclusively, are finite and continuous for all values of x between x_0 and $x_0 + h$; and if

$$\text{F}'(x_0) = 0, \quad \text{F}''(x_0) = 0, \ldots\ldots \text{F}^{n-1}(x_0) = 0,$$
$$f'(x_0) = 0, \quad f''(x_0) = 0, \ldots\ldots f^{n-1}(x_0) = 0,$$

and if, besides, $f'(x)$, $f''(x)$, ... $f^n(x)$ severally do not change sign between the limits, then

$$\frac{\text{F}(x_0 + h) - \text{F}(x_0)}{f(x_0 + h) - f(x_0)} = \frac{\text{F}^n(x_0 + \theta h)}{f^n(x_0 + \theta h)}. \tag{12}$$

COR. I.—Hence, if $\text{F}(x_0) = 0$, and $f(x_0) = 0$, (see Cor. I of last Art.)

$$\frac{\text{F}(x_0 + h)}{f(x_0 + h)} = \frac{\text{F}^n(x_0 + \theta h)}{f^n(x_0 + \theta h)}. \tag{13}$$

COR. II.—Also, if x_0, which is the inferior value of x, $= 0$, then, writing x for h, as h is measured from the origin, we have

$$\frac{\text{F}(x) - \text{F}(0)}{f(x) - f(0)} = \frac{\text{F}^n(\theta x)}{f^n(\theta x)}. \tag{14}$$

COR. III.—And if $\text{F}(0) = 0$, and $f(0) = 0$, then

$$\frac{\text{F}(x)}{f(x)} = \frac{\text{F}^n(\theta x)}{f^n(\theta x)}. \tag{15}$$

99.] In equation (12), which is the result including all others of the above Articles, let a specific form be assigned to $f(x)$, such as will satisfy the requisite conditions and render the equation applicable to future purposes; as for instance, let $(x) = (x-x_0)^n$, wherein n is positive and integral; then

$$\therefore\ f(x_0) = 0, \quad \text{and} \quad f(x_0+h) = h^n;$$

$$f'(x) = n(x-x_0)^{n-1}, \qquad f'(x_0) = 0,$$

$$f''(x) = n(n-1)(x-x_0)^{n-2}, \qquad f''(x_0) = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$f^{n-1}(x) = n(n-1)\ldots 3.2(x-x_0), \qquad f^{n-1}(x_0) = 0,$$

$$f^n(x) = n(n-1)\ldots 3.2.1, \qquad f^n(x_0) = n(n-1)(n-2)\ldots 3.2.1.$$

And as $f^n(x)$ does not involve x, it has the same value whatever x be, therefore

$$f^n(x_0+\theta h) = n(n-1)(n-2)\ldots\ldots 3.2.1;$$

and substituting these values in the general result, we have

$$\mathrm{F}(x_0+h) - \mathrm{F}(x_0) = \frac{h^n}{1.2.3\ldots(n-1)n}\,\mathrm{F}^n(x_0+\theta h); \tag{16}$$

the conditions to which this equation is subject being, that $\mathrm{F}(x)$, $\mathrm{F}'(x)$, $\mathrm{F}''(x)\ldots\ldots \mathrm{F}^n(x)$ are finite and continuous for all values of x between x_0 and x_0+h, and that $\mathrm{F}'(x_0)=0$, $\mathrm{F}''(x_0)=0$, up to $\mathrm{F}^{n-1}(x_0) = 0$; but that $\mathrm{F}^n(x_0)$ does not vanish.

100.] Cor. I.—Hence, if $\mathrm{F}(x_0) = 0$, we have

$$\mathrm{F}(x_0+h) = \frac{h^n}{1.2.3\ldots n}\,\mathrm{F}^n(x_0+\theta h); \tag{17}$$

and as a particular case of this, let $x_0 = 0$; then

$$\mathrm{F}(h) = \frac{h^n}{1.2.3\ldots n}\,\mathrm{F}^n(\theta h), \tag{18}$$

and writing x for h,

$$\mathrm{F}(x) = \frac{x^n}{1.2.3\ldots n}\,\mathrm{F}^n(\theta x); \tag{19}$$

which proposition may be enunciated as follows:

Theorem V.—If $\mathrm{F}(x)$ be a function of x, which, as also all its derived-functions up to the nth inclusively, are finite and

continuous for all values of x between 0 and x; and if $\mathrm{F}(x)$, $\mathrm{F}'(x)$, ... $\mathrm{F}^{n-1}(x)$ severally vanish when $x = 0$, then

$$\mathrm{F}(x) = \frac{x^n}{1.2.3\ldots n}\,\mathrm{F}^n(\theta x).$$

COR. II.—In equation (16) let $x_0 = 0$, then writing x for h,

$$\mathrm{F}(x) - \mathrm{F}(0) = \frac{x^n}{1.2.3\ldots n}\,\mathrm{F}^n(\theta x); \tag{20}$$

of which result (19) is a particular case.

101.] Let us consider some particular cases of these general equations which arise from given values of n.

In equation (16) suppose that $\mathrm{F}'(x_0)$ does not vanish, then $n = 1$, and

$$\mathrm{F}(x_0 + h) - \mathrm{F}(x_0) = h\,\mathrm{F}'(x_0 + \theta h); \tag{21}$$

which gives us a symbolical expression for the right-hand member of equation (7), Art. 18, including its residual expression R.

Or suppose again that $\mathrm{F}'(x_0) = 0$, but that $\mathrm{F}''(x_0)$ is finite, then $n = 2$, and

$$\mathrm{F}(x_0 + h) - \mathrm{F}(x_0) = \frac{h^2}{1.2}\,\mathrm{F}''(x_0 + \theta h). \tag{22}$$

Suppose again, in equation (20), that $\mathrm{F}(0) = 0$, but that $\mathrm{F}'(0)$ is finite, then

$$\mathrm{F}(x) = x\,\mathrm{F}'(\theta x); \tag{23}$$

and therefore every function of a variable x, which vanishes when $x = 0$, has x for a factor, unless the first-derived-function is equal to ∞, when $x = 0$.

Thus, for example, let $\mathrm{F}(x) = \sin x$, which $= 0$ if $x = 0$; then its derived-function $= \cos x$, which $= 1$, when $x = 0$; therefore x is a factor of $\sin x$.

Similarly, if $\mathrm{F}(x) = \tan x$, which $= 0$, if $x = 0$; then $\mathrm{F}'(x) = (\sec x)^2$, which $= 1$, when $x = 0$; therefore x is a factor of $\tan x$.

But if $\mathrm{F}(x) = e^{-\frac{1}{x^2}}$, which $= 0$, if $x = 0$, $\mathrm{F}'(x) = \frac{2}{x^3}e^{-\frac{1}{x^2}}$, which $= \frac{0}{0}$, when $x = 0$, and is therefore indeterminate; whence we cannot conclude that x is a factor of $e^{-\frac{1}{x^2}}$.

The statement made in most of the ordinary text books on the Theory of Algebraical Expressions, that, if $F(x) = 0$, when $x = 0$, x is a factor of $F(x)$: or (what is equivalent) that, if $F(x) = 0$, when $x = a$, $x - a$ is a factor of $F(x)$: is only one of too many *universal* propositions which are unduly assumed; it may be and is true in most cases, but it is an unphilosophical desire of generalization which will lead us to conclude that it is true in *all* cases. All possible functions cannot be known; and therefore a conscious ignorance ought to protect us against such an error, and be a much more cogent reason against it than the knowledge of a particular instance which disproves the induction; practically however it is found not to be so; let the student therefore be on his guard against such universal assumptions, which rest for proof only on human ignorance.

CHAPTER V.

ON THE DETERMINATION OF THE ORDERS OF INFINITESIMALS, AND THE EVALUATION OF QUANTITIES WHICH FOR PARTICULAR VALUES OF THE VARIABLES ASSUME THE FORMS:

$$\frac{0}{0},\ \frac{\infty}{\infty},\ 0\times\infty,\ \infty-\infty,\ 0^0,\ \infty^0,\ 1^{\infty},\ 0^{\infty}.$$

102.] In questions such as those proposed for discussion in the present Chapter, it is important to observe what is the exact meaning of the numerical unit; namely, that it is *the ratio of equality*, and independent of the particular magnitude of the quantities which are compared. Hence it follows, that it is immaterial whether the quantities are infinitesimal, finite, or infinite, provided that we can assure ourselves that they are equal. Whenever therefore the same factor is involved in the numerator and denominator of a fraction, be it of any magnitude and kind, it may be divided out, and the value of the function will not be changed by the division: by this means expressions on which operations are to be performed can often be simplified; an instance will illustrate the process and its effect. Suppose it is required to determine the value of

$$\frac{x-a+(2x^2-2ax)^{\frac{1}{2}}}{(x^2-a^2)^{\frac{1}{2}}},$$

when $x = a$; in which case the fraction assumes the form $\frac{0}{0}$; but why? Because both numerator and denominator involve a factor $(x-a)^{\frac{1}{2}}$, which is equal to 0 when $x = a$; but the factor in the numerator being exactly equal to the factor in the denominator, the ratio of one to the other is unity; by which therefore $\frac{(x-a)^{\frac{1}{2}}}{(x-a)^{\frac{1}{2}}}$ may be replaced, and the fraction becomes

$$\frac{(x-a)^{\frac{1}{2}}+(2x)^{\frac{1}{2}}}{(x+a)^{\frac{1}{2}}},$$

which is equal to 1, when $x = a$.

SECTION 1.—*On the Determination of Orders of Infinitesimals.*

103.] It is well first to repeat with greater exactness the main points of the account of infinitesimals which was given in rough outline in Art. 8 and 9, Chapter I.

The determination of the Order of Infinitesimals (which is a relative term) requires a standard to compare them with. This standard is called the Base of Infinitesimals; and according to the power of the base, which a given infinitesimal expression involves, is its order called.

Thus suppose i to be the base, then ($a, b, c, \ldots\ldots$ being finite numbers) ai, bi^3, ci^n, are respectively infinitesimals of the first, third, nth orders; $i^{\frac{1}{2}}$, $i^{\frac{3}{4}}$ ...... are infinitesimals respectively of the one-half and three-fourths orders. Similarly may there be infinitesimals of negative orders.

Suppose that $\text{F}(i)$ is a function of infinitesimals, and an infinitesimal itself, whose order is to be compared with i as the base. If $\text{F}(i)$ is an infinitesimal of the nth order, i^n is a factor of it, and of all the powers of i that enter as factors into its composition, i^n is the lowest: higher ones being neglected of necessity by virtue of Theorem VI, Art. 9. Therefore, if $\text{F}(i)$ be divided by i^n, the quotient is finite; but if it be divided by i, raised to any index lower than n, the quotient is infinitesimal; and if it be divided by i, raised to any index higher than n, the quotient is infinite. Hence we are enabled to construct the following definition of the order of an infinitesimal expression:

DEF.—*Suppose* $\text{F}(\text{i})$ *to be the infinitesimal expression whose order is to be determined; then, if* $\frac{\text{F}(\text{i})}{\text{i}^r}$ *is infinitesimal for all values of* r *less than* n, *and infinite for all values of* r *greater than* n, $\text{F}(\text{i})$ *is an infinitesimal of the* n*th order.*

Hence every finite quantity is an infinitesimal of the order 0; for suppose k to be finite, then $\frac{k}{i^r}$ is infinitesimal for all values of r less than 0, and infinity for all values of r greater than 0.

Hence, if we use 0 for the general symbol of an infinitesimal, the form, which all finite quantities assume from this point of view, is 0^0. This is also evident from the example given in the

last Article, where both numerator and denominator $= 0$, when $x = a$; so that $x - a$ is an infinitesimal which enters into both numerator and denominator, whereby the fraction is of the form $\frac{k \times 0}{k' \times 0}$, k and k' being constants; and as the infinitesimals, being exactly even, are of the same order, the form of the fraction is $0^0 \frac{k}{k'}$; and as 0^0 is equal to 1 in this case, the true value of the fraction is $\frac{k}{k'}$.

104.] Writing i for x in equation (19), Art. 100, we have

$$\text{F}(i) = \frac{i^n}{1.2.3\ldots(n-1)n}\,\text{F}^n(\theta i). \tag{1}$$

Now observe the conditions to which $\text{F}(i)$ is subject; viz. that

$$\text{F}(0) = 0, \; \text{F}'(0) = 0, \ldots\ldots \text{F}^{n-1}(0) = 0,$$

and that $\text{F}^n(i)$ is the derivative, which first does not vanish when $i = 0$; and that $\text{F}(i)$ and all its derivatives up to the nth inclusively are continuous and finite for all values of i between i and 0; and observe also the limits, viz. i and 0; but that, as we have to use the equation only when $i = 0$, we may make the difference between the limits to be infinitesimal. Hence

$$\frac{\text{F}(i)}{i^r} = \frac{i^{n-r}}{1.2.3\ldots(n-1)n}\,\text{F}^n(\theta i), \tag{2}$$

which is infinitesimal for all values of r less than n, and infinite for all values of r greater than n; and therefore we have the following Theorem for the determination of the order of infinitesimals.

THEOREM.—If $\text{F}(i)$ be a function of an infinitesimal i, such that $\text{F}(0) = 0$, and that all its derived-functions up to the nth vanish when $i = 0$, but that $\text{F}^n(i)$ does not vanish, then $\text{F}(i)$ is an infinitesimal of the nth order, if i be taken as the infinitesimal base.

If, when $i = 0$, $\text{F}(i)$ assumes an indeterminate form, its value must be found by the methods which are explained in the following sections of this Chapter.

In all the following examples i is taken as the base of infinitesimals.

Ex. 1. To determine the order of infinitesimal of $\sin i$.

$$\text{F}(i) = \sin i = 0, \text{ when } i = 0,$$

$$\text{F}'(i) = \cos i = 1, \ldots\ldots$$

$\therefore$ $\sin i$ is an infinitesimal of the first order.

Ex. 2. Find the order of infinitesimal of $\tan i - \sin i$.

$$\text{F}(i) = \tan i - \sin i = 0, \text{ when } i = 0,$$

$$\text{F}'(i) = (\sec i)^2 - \cos i = 0, \ldots\ldots$$

$$\text{F}''(i) = 2(\sin i)^2 \tan i + \sin i = 0, \text{ when } i = 0,$$

$$\text{F}'''(i) = 6(\sec i)^4\, 4(\sec i)^2 + \cos i = 3, \text{ when } i = 0.$$

Therefore $\tan i - \sin i$ is a function of an infinitesimal which vanishes when $i = 0$, and so do all its derived-functions up to the third, which does not vanish; therefore $\tan i - \sin i$ is an infinitesimal of the third order, when i is the base.

Ex. 3. To determine the order of $e^i - 2\sin i - e^{-i}$.

$$\text{F}(i) = e^i - 2\sin i - e^{-i} = 0, \text{ when } i = 0,$$

$$\text{F}'(i) = e^i - 2\cos i + e^{-i} = 0, \ldots\ldots$$

$$\text{F}''(i) = e^i + 2\sin i - e^{-i} = 0, \ldots\ldots$$

$$\text{F}'''(i) = e^i + 2\cos i + e^{-i} = 4. \ldots\ldots$$

Therefore $e^i - 2\sin i - e^{-i}$ is an infinitesimal of the fourth order.

Ex. 4. To determine the order of infinitesimal of $e^{i^2} - 1$.

$$\text{F}(i) = e^{i^2} - 1 = 0, \text{ when } i = 0,$$

$$\text{F}'(i) = 2\,i\,e^{i^2} = 0, \ldots\ldots$$

$$\text{F}''(i) = 4\,i^2 e^{i^2} + 2\,e^{i^2} = 2, \ldots\ldots$$

$\therefore$ $e^{i^2} - 1$ is an infinitesimal of the second order.

Similarly let the student prove that $\tan i$, $\sin^{-1} i$, $e^i - e^{-i}$, are infinitesimals of the first order; that $\text{versin}\, i$, $\sin i^2$ are infinitesimals of the second order; that $e^{-\frac{1}{i}}$ is an infinitesimal of the order ∞.

105.] Similarly, if $\text{F}(x)$ be a function of x which vanishes when $x = a$, replacing $x - a$ by i, the function becomes $\phi(i)$, which vanishes when $i = 0$, and its order of infinitesimal may be determined in the manner above explained.

SECTION 2.—*Evaluation of quantities of the forms*

$$\frac{0}{0},\quad \frac{\infty}{\infty},\quad 0\times\infty,\quad \infty-\infty.$$

106.] Evaluation of quantities of the form $\frac{0}{0}$.

If a quotient of two functions of a variable or variables, for particular values of these, assumes the form $\frac{0}{0}$, it is plain that such is the case only because certain factors in the numerator and denominator become 0 for these particular values, that is, become infinitesimals. It is plain too from what has been said, (see Theorem IV, Art. 10,) that, if these infinitesimals are of the same order, the fraction will be a finite quantity; and, if that in the numerator be of a higher order than that in the denominator, the value of the fraction is 0; and, if that in the denominator be of a higher order than that in the numerator, the fraction is ∞: an example will render this plain.

Suppose we have to evaluate $\frac{(x-a)^m \text{M}}{(x-a)^n \text{N}}$ when $x=a$, M and N being functions of x or not, as the case may be, but not involving any factor of the form $(x-a)$, and not vanishing when $x=a$; the fraction assumes the form $\frac{0}{0}$; but the law of indices authorizes us to put it under the form

$$(x-a)^{m-n}\frac{\text{M}}{\text{N}};$$

and if $x=a$, it $=0$, if m be $>n$,

$$. \quad . \quad . \quad . \quad . \quad . \quad = \frac{\text{M}}{\text{N}},\ \text{if } m=n,$$

$$. \quad . \quad . \quad . \quad . \quad . \quad = \infty,\ \text{if } m \text{ be} < n.$$

It will be seen from Article 110 that similar results are true, if the numerator and denominator be infinities.

107.] Hence then the first step towards the evaluation of such quantities is to detect, if possible, the factors common to both the numerator and the denominator, and to divide them out, and then to evaluate the resulting fraction by giving to

the variables the assigned values; of which method some examples are subjoined.

Ex. 1.

$$\frac{x^3-1}{x^3-2x^2+2x-1} = \frac{0}{0}, \text{ when } x = 1,$$

$$= \frac{(x^2+x+1)(x-1)}{(x^2-x+1)(x-1)} = \frac{x^2+x+1}{x^2-x+1} = 3, \text{ when } x = 1.$$

Ex. 2.

$$\frac{(a^2-x^2)^{\frac{1}{2}}+(a-x)}{(a-x)^{\frac{1}{2}}+(a^3-x^3)^{\frac{1}{2}}} = \frac{0}{0},$$

if $x = a$; the common factor is $(a-x)^{\frac{1}{2}}$, divide numerator and denominator by it, and we have

$$\frac{(a+x)^{\frac{1}{2}}+(a-x)^{\frac{1}{2}}}{1+(a^2+ax+x^2)^{\frac{1}{2}}} = \frac{(2a)^{\frac{1}{2}}}{1+a3^{\frac{1}{2}}}, \text{ if } x = a.$$

108.] The preceding method of evaluating indeterminate forms may often be advantageously employed in one or other of the two following ways.

Firstly, expand by the Binomial Theorem, or by Maclaurin's Series, or some other equivalent method, the given functions in terms of the infinitesimal, and then by dividing out the common factors, the function will assume a determinate form.

Ex. 1. Determine the value of $\frac{e^x-e^{-x}}{\sin x}$, when $x = 0$.

$$\frac{e^x-e^{-x}}{\sin x} = \frac{1+\frac{x}{1}+\frac{x^2}{1.2}+\ldots\ldots-\left(1-\frac{x}{1}+\frac{x^2}{1.2}-\ldots\ldots\right)}{x-\frac{x^3}{1.2.3}+\ldots\ldots},$$

$$= \frac{2\left(x+\frac{x^3}{1.2.3}+\ldots\ldots\right)}{x-\frac{x^3}{1.2.3}+\ldots\ldots},$$

$$= \frac{2\left(1+\frac{x^2}{1.2.3}+\ldots\ldots\right)}{1-\frac{x^2}{1.2.3}+\ldots\ldots},$$

$$= 2, \text{ when } x = 0.$$

Secondly, if the function assumes an indeterminate form when $x = a$, write h for $x - a$, or $a - x$ (as may be more convenient), that is, for x substitute $a + h$, or $a - h$, in all the terms of the function; and develope them in a series of ascending powers of h, and after cancelling, by means of division, the common powers of h, put $h = 0$, and the result is the determinate value of the function.

Ex. 1. Evaluate $\dfrac{x-a+(2ax-2a^2)^{\frac{1}{2}}}{(x^2-a^2)^{\frac{1}{2}}}$.

For x write $a + h$, and the function becomes

$$\frac{a+h-a+(2a^2+2ah-2a^2)^{\frac{1}{2}}}{\{(a+h)^2-a^2\}^{\frac{1}{2}}},$$

$$= \frac{h+(2ah)^{\frac{1}{2}}}{(2ah+h^2)^{\frac{1}{2}}},$$

$$= \frac{h^{\frac{1}{2}}+(2a)^{\frac{1}{2}}}{(2a+h)^{\frac{1}{2}}} = 1, \text{ when } h = 0.$$

109.] In cases however where it is difficult to detect the common factors, as well as in all cases where the necessary conditions are fulfilled, the theorems of the preceding Chapter enable us to evaluate these quantities.

Suppose $f(x)$ and $\phi(x)$ to be two functions of x, which $= 0$ when $x = x_0$. Suppose also that, when $x = x_0$, their several derived-functions up to the $(n-1)$th inclusively vanish, but that the nth neither vanishes nor becomes infinite: then, if $f(x)$ and $\phi(x)$ are finite and continuous for all values of x between x_0 and $x_0 + h$, by the theorem contained in equation (17), Art. 100,

$$f(x_0+h) = \frac{h^n}{1.2.3\ldots n} f^n(x_0+\theta h),$$

$$\phi(x_0+h) = \frac{h^n}{1.2.3\ldots n} \phi^n(x_0+\theta h),$$

$$\therefore \frac{f(x_0+h)}{\phi(x_0+h)} = \frac{f^n(x_0+\theta h)}{\phi^n(x_0+\theta h)}.$$

Suppose now that h, the difference between the superior and

inferior limits, becomes infinitesimal; then neglecting h when added to x_0, we have

$$\frac{f(x_0)}{\phi(x_0)} = \frac{f^n(x_0)}{\phi^n(x_0)}; \tag{3}$$

that is, the value of the ratio $\frac{f(x_0)}{\phi(x_0)}$, which presents itself under the indeterminate form $\frac{0}{0}$, is the ratio of the values of the derived-functions of the numerator and denominator, which are the first not to vanish, when $x = x_0$.

Of this general proposition the following are particular instances:

Suppose the functions themselves to vanish, but not their first-derived, then

$$\frac{f(x_0)}{\phi(x_0)} = \frac{f'(x_0)}{\phi'(x_0)}; \tag{4}$$

and if the functions themselves and also their first-derived vanish, but not the second-derived, then

$$\frac{f(x_0)}{\phi(x_0)} = \frac{f''(x_0)}{\phi''(x_0)}, \tag{5}$$

and so on.

If the same order of derived-functions do not simultaneously vanish, then the indeterminate form has for its value either 0 or ∞, according as the denominator or numerator has first ceased to vanish. This is plain from the proof we have given; because h will be of different powers in the equivalents of $f(x_0 + h)$ and $\phi(x_0 + h)$, and therefore will not disappear in the ratio.

Ex. 1. Evaluate $\frac{x^3-1}{x^3-2x^2+2x-1}$, when $x = 1$.

$$\frac{f(x)}{\phi(x)} = \frac{x^3-1}{x^3-2x^2+2x-1} = \frac{0}{0}, \text{ when } x = 1,$$

$$= \frac{f'(x)}{\phi'(x)} = \frac{3x^2}{3x^2-4x+2} = 3, \text{ when } x = 1.$$

Ex. 2. Evaluate $\frac{e^x - e^{-x}}{\sin x}$, when $x = 0$.

$$\frac{f(x)}{\phi(x)} = \frac{e^x - e^{-x}}{\sin x} = \frac{0}{0}, \text{ when } x = 0.$$

$$= \frac{f'(x)}{\phi'(x)} = \frac{e^x + e^{-x}}{\cos x} = 2, \text{ when } x = 0.$$

Ex. 3. Evaluate $\frac{1-\cos x}{x^2}$, when $x = 0$.

$$\frac{f(x)}{\phi(x)} = \frac{1-\cos x}{x^2} = \frac{0}{0}, \text{ when } x = 0,$$

$$= \frac{f'(x)}{\phi'(x)} = \frac{\sin x}{2x} = \frac{0}{0}, \quad . \quad . \quad . \quad .$$

$$= \frac{f''(x)}{\phi''(x)} = \frac{\cos x}{2} = \frac{1}{2}, \text{ when } x = 0.$$

Ex. 4. Evaluate $\frac{e^x - 2\sin x - e^{-x}}{x-\sin x}$, when $x = 0$.

$$\frac{f(x)}{\phi(x)} = \frac{e^x - 2\sin x - e^{-x}}{x-\sin x} = \frac{0}{0}, \text{ when } x = 0,$$

$$= \frac{f'(x)}{\phi'(x)} = \frac{e^x - 2\cos x + e^{-x}}{1-\cos x} = \frac{0}{0}, \quad . \quad . \quad . \quad .$$

$$= \frac{f''(x)}{\phi''(x)} = \frac{e^x + 2\sin x - e^{-x}}{\sin x} = \frac{0}{0}, \quad . \quad . \quad . \quad .$$

$$= \frac{f'''(x)}{\phi'''(x)} = \frac{e^x + 2\cos x + e^{-x}}{\cos x} = 4, \quad . \quad . \quad . \quad .$$

Ex. 5. Evaluate $\frac{\log x}{x-1}$, when $x = 1$.

$$\frac{f(x)}{\phi(x)} = \frac{\log x}{x-1} = \frac{0}{0}, \text{ when } x = 1,$$

$$= \frac{f'(x)}{\phi'(x)} = \frac{1}{x} = 1, \quad . \quad . \quad . \quad .$$

Ex. 6. Evaluate $\frac{(x-a)^n}{e^x - e^a}$, when $x = a$.

$$\frac{f(x)}{\phi(x)} = \frac{(x-a)^n}{e^x - e^a} = \frac{0}{0}, \text{ when } x = a,$$

$$= \frac{f'(x)}{\phi'(x)} = \frac{n(x-a)^{n-1}}{e^x},$$

which, when $x = a$, is either 0, e^{-a}, or ∞, according as n is greater than, equal to, or less than, 1.

On examination of the above examples, it will be seen that the result is infinitesimal, finite or infinite according as the order of infinitesimals in the numerator, determined by the method of Section 1 of this Chapter, is higher than, the same as, or lower than, the order of the denominator.

110.] Evaluation of indeterminate quantities of the form $\frac{\infty}{\infty}$.

Let $f(x)$ and $\phi(x)$ be two functions of x, which become infinite when $x = x_0$, then, as their reciprocals become zero, the ratio of the functions may be evaluated by the former method, as follows:

$$\frac{f(x)}{\phi(x)} = \frac{\infty}{\infty}, \text{ when } x = x_0,$$

$$= \frac{\frac{1}{\phi(x)}}{\frac{1}{f(x)}} = \frac{0}{0}, \quad . \quad . \quad . \quad .$$

$$= \frac{\frac{\phi'(x)}{\{\phi(x)\}^2}}{\frac{f'(x)}{\{f(x)\}^2}} = \frac{\phi'(x)}{f'(x)} \frac{\{f(x)\}^2}{\{\phi(x)\}^2};$$

whence, dividing out common terms, we have, when $x = x_0$,

$$\frac{f(x_0)}{\phi(x_0)} = \frac{f'(x_0)}{\phi'(x_0)};$$

If the first-derived-functions $f'(x_0)$, $\phi'(x_0)$ also become infinite, they must be evaluated in the same way as $\frac{f(x_0)}{\phi(x_0)}$, and we shall have

$$\frac{f(x_0)}{\phi(x_0)} = \frac{f''(x_0)}{\phi''(x_0)};$$

and if the several derived-functions vanish or (if it be possible) become infinite up to the nth, when $x = x_0$, and the nth are finite,

$$\frac{f(x_0)}{\phi(x_0)} = \frac{f'(x_0)}{\phi'(x_0)} = \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad = \frac{f^n(x_0)}{\phi^n(x_0)}.$$

The determinate value therefore of such indeterminate functions is the ratio of those derived-functions of the numerator and denominator which are the first to become finite when $x = x_0$.

Ex. 1. Evaluate $\frac{\log x}{\cot x}$, when $x = 0$.

$$\frac{f(x)}{\phi(x)} = \frac{\log x}{\cot x} = \frac{-\infty}{\infty}, \text{ when } x = 0,$$

$$= \frac{f'(x)}{\phi'(x)} = -\frac{(\sin x)^2}{x} = \frac{0}{0}, \quad \ldots \ldots$$

$$= -\frac{2 \sin x \cos x}{1}, \text{ by last Article,}$$

$$= 0, \text{ when } x = 0;$$

$$\therefore \quad \frac{\log x}{\cot x} = 0, \text{ when } x = 0.$$

Ex. 2. Evaluate $\frac{1-\log x}{e^{\frac{1}{x}}}$, when $x = 0$.

$$\frac{f(x)}{\phi(x)} = \frac{1-\log x}{e^{\frac{1}{x}}} = \frac{\infty}{\infty}, \text{ when } x = 0.$$

$$= \frac{f'(x)}{\phi'(x)} = \frac{\frac{1}{x}}{\frac{1}{x^2} e^{\frac{1}{x}}} = \frac{x}{e^{\frac{1}{x}}} = \frac{0}{\infty} = 0, \text{ when } x = 0.$$

Ex. 3. Evaluate $\frac{x^n}{e^x}$, when $x = \infty$.

$$\frac{f(x)}{\phi(x)} = \frac{x^n}{e^x} = \frac{\infty}{\infty}, \text{ when } x = \infty,$$

$$= \frac{f'(x)}{\phi'(x)} = \frac{n(x)^{n-1}}{e^x} = \frac{\infty}{\infty}, \quad \ldots \ldots$$

$$= \ldots \ldots \ldots \ldots \ldots \ldots$$

$$= \frac{f^n(x)}{\phi^n(x)} = \frac{n(n-1)(n-2)\ldots 3.2.1}{e^x} = 0, \text{ when } x = \infty.$$

The result of which shews that e^x is, when $x = \infty$, an infinity of a higher order than x^n; also a similar property is true of a^x; a finite quantity therefore raised to an infinite power is an infinity of a higher order than an infinite quantity raised to a finite power.

Ex. 4. Evaluate $\dfrac{e^{-\frac{1}{x^2}}}{x^{2n}}$, when $x = 0$.

For x^2 write $\dfrac{1}{z}$, $\therefore$ $z = \infty$, when $x = 0$;

and the function becomes

$$\frac{z^n}{e^z} = \frac{\infty}{\infty}, \text{ when } z = \infty,$$

$$= \frac{nz^{n-1}}{e^z} = \frac{\infty}{\infty}, \ldots$$

$$= \frac{n(n-1)z^{n-2}}{e^z} = \frac{\infty}{\infty}, \ldots$$

$$= \ldots\ldots\ldots$$

$$= \frac{n(n-1)\ldots 3.2.1}{e^z} = 0, \text{ when } z = \infty;$$

therefore the function is equal to 0, when $x = 0$; whence we conclude that $e^{-\frac{1}{x^2}}$ is an infinitesimal of a higher order than x^{2n}, however large be the value of n.

This is a result of some importance, insomuch as it shews that although $e^{-\frac{1}{x^2}}$ vanishes when $x = 0$, whence it might be inferred that x is a factor of it, yet it admits of no factor of the form x^{2n} however large n be. Compare equation (23), Art. 101, and the remarks thereupon.

Ex. 5. Evaluate $\dfrac{\log x}{x}$, when $x = \infty$.

$$\frac{\log x}{x} = \frac{\infty}{\infty}, \text{ when } x = \infty,$$

$$= \frac{1}{x} = 0, \text{ when } x = \infty;$$

if x therefore becomes infinite, it is an infinity of a higher order than its logarithm.

111.] Evaluation of functions which for particular values assume the indeterminate form $0 \times \infty$.

Let $f(x)$ and $\phi(x)$ be two functions of x which respectively become 0 and ∞, when $x = x_0$; then their product may be put under the form

$$\frac{f(x)}{\dfrac{1}{\phi(x)}} \text{ which } = \frac{0}{0}, \text{ when } x = x_0,$$

and may be evaluated according to the method of the last two Articles.

Ex. 1. Evaluate $e^{-x}\log x$, when $x = \infty$.

$$e^{-x}\log x = \frac{\log x}{e^x} = \frac{\infty}{\infty}, \text{ when } x = \infty,$$

$$= \frac{1}{xe^x} = 0, \text{ when } x = \infty;$$

$$\therefore \quad e^{-x}\log x = 0, \text{ when } x = \infty.$$

Ex. 2. Evaluate $x^n \log x$, when $x = 0$.

$$x^n \log x = \frac{\log x}{x^{-n}} = \frac{1}{-nx^{-n}} = \frac{x^n}{-n} = 0, \text{ when } x = 0;$$

$$\therefore \quad x^n \log x = 0, \text{ when } x = 0.$$

112.] Evaluation of functions which for particular values assume the form
$$\infty - \infty.$$

Let $f(x)$ and $\phi(x)$ be two functions of x, which $= 0$ when $x = x_0$, in which case $\dfrac{1}{f(x)}$ and $\dfrac{1}{\phi(x)}$ are both $= \infty$; then

$$\frac{1}{f(x_0)} - \frac{1}{\phi(x_0)} = \frac{\phi(x_0) - f(x_0)}{\phi(x_0)\,f(x_0)} = \frac{0}{0},$$

and may be evaluated as heretofore.

Ex. 1. Evaluate $\dfrac{2}{x^2-1} - \dfrac{1}{x-1}$, when $x = 1$.

$$\frac{2}{x^2-1} - \frac{1}{x-1} = \frac{2-x-1}{x^2-1} = \frac{0}{0}, \text{ when } x = 1,$$

$$= \frac{-1}{2x} = -\frac{1}{2}, \text{ when } x = 1.$$

Ex. 2. Evaluate $\sec x - \tan x$, when $x = \frac{\pi}{2}$.

$$\sec x - \tan x = \infty - \infty, \text{ when } x = \frac{\pi}{2},$$

$$= \frac{1-\sin x}{\cos x} = \frac{0}{0}, \quad . \quad . \quad . \quad .$$

$$= \frac{-\cos x}{-\sin x} = 0, \text{ when } x = \frac{\pi}{2}.$$

Therefore, when $x = \frac{\pi}{2}$, $\sec x$ and $\tan x$ are either absolutely equal, or differ by a quantity which must be neglected in their algebraical sum.

Ex. 3. Evaluate $\frac{1}{\log x} - \frac{x}{\log x}$, when $x = 1$.

$$\frac{1}{\log x} - \frac{x}{\log x} = \frac{1-x}{\log x} = \frac{0}{0}, \text{ when } x = 1,$$

$$= -x = -1, \text{ when } x = 1.$$

SECTION 3.—*Evaluation of Explicit Functions of a single variable, which for particular values assume the forms* 0^0, ∞^0, 1^∞, 0^∞.

113.] Let $f(x)$ and $\phi(x)$ be two functions of x which, when $x = x_0$, assume such values that $f(x_0)^{\phi(x_0)}$ is of one or other of the above forms.

$$\text{Let} \qquad y = f(x)^{\phi(x)},$$

$$\therefore \quad \log_e y = \phi(x) \log_e f(x);$$

and as $\log f(x)$ has singular values when $f(x) = 0$, or $= 1$, or $= \infty$, we may express the last equation in the form

$$\log_e y = \frac{\log_e f(x)}{\frac{1}{\phi(x)}},$$

which, for the particular value of x_0, will be of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, and may be evaluated according to the methods explained in the last Section.

Ex. 1. Evaluate x^x, when $x = 0$.

$$\text{Let} \qquad y = x^x, \qquad \therefore \quad \log_e y = x \log_e x;$$

and by Article 111, Ex. 2,

$$x \log_e x = 0, \qquad \therefore \quad \log_e y = 0;$$

$$\therefore \quad y = x^x = 1, \text{ when } x = 0.$$

Ex. 2. Evaluate $x^{\frac{1}{x}}$, when $x = \infty$.

Let $$y = x^{\frac{1}{x}},$$

$$\therefore \quad \log y = \frac{\log x}{x} = \frac{\infty}{\infty}, \text{ when } x = \infty;$$

$$= \frac{1}{x} = 0, \quad . \quad . \quad . \quad .$$

$$\therefore \quad y = x^{\frac{1}{x}} = 1, \text{ when } x = \infty.$$

Ex. 3. Evaluate $(1+ax)^{\frac{b}{x}}$, when $x = 0$, and when $x = \infty$.

Let $$y = (1+ax)^{\frac{b}{x}},$$

$$\therefore \quad \log y = \frac{b \log (1+ax)}{x} = \frac{0}{0}, \text{ when } x = 0,$$

$$= \frac{ab}{1+ax} = ab, \quad . \quad . \quad . \quad .$$

$$\therefore \quad y = (1+ax)^{\frac{b}{x}} = e^{ab}, \text{ when } x = 0.$$

Again, $$\log y = \frac{b \log (1+ax)}{x} = \frac{\infty}{\infty}, \text{ when } x = \infty,$$

$$= \frac{ab}{1+ax} = 0, \text{ when } x = \infty,$$

$$\therefore \quad (1+ax)^{\frac{b}{x}} = 1, \text{ when } x = \infty.$$

Section 4.—*Evaluation of Functions of two variables, which for particular values of the variables assume indeterminate forms.*

114.] Let $u = \text{F}(x, y) = c$; whence we have

$$\frac{dy}{dx} = -\frac{\left(\frac{d\text{F}}{dx}\right)}{\left(\frac{d\text{F}}{dy}\right)};$$

Suppose that x_0 and y_0 are values of x and y, which simultaneously satisfy the given equation, and are such that

$\left(\frac{dF}{dx}\right) = 0$, $\left(\frac{dF}{dy}\right) = 0$, then $\frac{dy}{dx}$ is of the form $\frac{0}{0}$, and must be evaluated according to the methods explained above, by taking the total differentials of the numerator and of the denominator; thus

$$\frac{dy}{dx} = -\frac{\left(\frac{dF}{dx}\right)}{\left(\frac{dF}{dy}\right)} = \frac{0}{0}, \text{ when } x = x_0 \text{ and } y = y_0,$$

$$= -\frac{\left(\frac{d^2F}{dx^2}\right) dx + \left(\frac{d^2F}{dx\,dy}\right) dy}{\left(\frac{d^2F}{dy^2}\right) dy + \left(\frac{d^2F}{dx\,dy}\right) dx},$$

and if the second partial derived-functions vanish for the particular values, then we must differentiate again numerator and denominator, and so on until the definite result is arrived at; but as the process in its most general form will have a closer consideration in a future part of the work, we will now do no more than give some examples illustrative of it.

Ex. 1. To find the values of $\frac{dy}{dx}$, when $x = 0$ and $y = 0$, having given

$$F(x, y) = ay^2 - x^3 - bx^2 = 0;$$

$$\therefore \quad \left(\frac{dF}{dx}\right) = -3x^2 - 2bx, \qquad \left(\frac{dF}{dy}\right) = 2ay;$$

$$\therefore \quad \frac{dy}{dx} = \frac{3x^2 + 2bx}{2ay} = \frac{0}{0}, \text{ when } x = 0 \text{ and } y = 0.$$

Taking the total differentials of the numerator and denominator, and dividing by dx,

$$\frac{dy}{dx} = \frac{6x + 2b}{2a\frac{dy}{dx}} = \frac{b}{a\frac{dy}{dx}}, \text{ when } x = y = 0;$$

$$\therefore \quad \left(\frac{dy}{dx}\right)^2 = \frac{b}{a}, \text{ and } \frac{dy}{dx} = \pm \left(\frac{b}{a}\right)^{\frac{1}{2}}.$$

Ex. 2. To determine the values of $\frac{dy}{dx}$, corresponding to $x = 0$ and $y = 0$, having given

$$x^4 + ay^3 - 2axy^2 - 3ax^2y = 0.$$

Differentiating as before, we find

$$\frac{dy}{dx} = \frac{4x^3 - 2ay^2 - 6axy}{3ax^2 + 4axy - 3ay^2} = \frac{0}{0}, \text{ when } x = y = 0,$$

$$= \frac{12x^2 - 4ay\frac{dy}{dx} - 6ay - 6ax\frac{dy}{dx}}{6ax + 4ax\frac{dy}{dx} + 4ay - 6ay\frac{dy}{dx}} = \frac{0}{0}, \text{ when } x = y = 0.$$

And here it is to be remarked; that in the next differentiation $\frac{dy}{dx}$ is to be considered constant; for although $\frac{dy}{dx}$ may have many values corresponding to the particular values of x and y, yet these values do not vary with small variations of x and y, and therefore are to be considered invariable when x and y are differentiated.

$$\therefore \quad \frac{dy}{dx} = \frac{24x - 4a\frac{dy^2}{dx^2} - 6a\frac{dy}{dx} - 6a\frac{dy}{dx}}{6a + 4a\frac{dy}{dx} + 4a\frac{dy}{dx} - 6a\frac{dy^2}{dx^2}},$$

$$= \frac{-4a\frac{dy^2}{dx^2} - 12a\frac{dy}{dx}}{6a + 8a\frac{dy}{dx} - 6a\frac{dy^2}{dx^2}}, \text{ when } x = y = 0,$$

$\therefore$ multiplying and reducing,

$$\frac{dy^3}{dx^3} - 2\frac{dy^2}{dx^2} - 3\frac{dy}{dx} = 0;$$

$$\therefore \quad \frac{dy}{dx} = 0 \text{ and } = 3 \text{ and } = -1.$$

A method similar to the preceding must be employed when the values of the variables render $\left(\frac{dF}{dx}\right) = \infty$, and $\left(\frac{dF}{dy}\right) = \infty$.

CHAPTER VI.

ON EXPANSION OF FUNCTIONS.

SECTION 1.—*On Functions of one Variable.*

115.] THE Theorems of Chapter IV afford rigorous proofs and limits of application of Taylor's and Maclaurin's Theorems, of the truth of which little more than a favourable presumption can be raised from what is said in Chapter III.

On referring to Art. 17, equation (3), and to Art. 18, equation (6), it will be seen, that if dx be an infinitesimal increment of x,

$$F(x+dx) - F(x) = F'(x)\,dx. \qquad (1)$$

But if Δx or (as we shall, to preserve an uniform notation, write) h, be finite, see Art. 18,

$$F(x+h) - F(x) = F'(x)\,h + R_1, \qquad (2)$$

writing R_1 for Rh; R_1 being therefore *a function of h*, which must be neglected when h is infinitesimal, but has a value finite and to be determined when h is finite. Or, in other words,

Given that $F(x)$ is continuous and finite for all values of x between x and $x+h$, it is required to expand $F(x+h)$ in a series of ascending powers of h.

We may also thus arrive at the equation (2) above.

If $F'(x)$ does not vanish, and remains finite and continuous for all values of x between x and $x+h$, then by equation (21), Art. 101,

$$F(x+h) - F(x) = h\,F'(x+\theta h),$$

which may be written in the form

$$F(x+h) - F(x) = h\,F'(x) + R_1.$$

We proceed to determine R_1:

$$R_1 = F(x+h) - F(x) - h\,F'(x),$$

therefore R_1 is a function of h which is equal to 0, when $h = 0$; also

$$\frac{dR_1}{dh} = F'(x+h) - F'(x) = 0, \text{ when } h = 0,$$

$$\frac{d^2R_1}{dh^2} = F''(x+h), \tag{3}$$

which does not vanish when $h = 0$.

Now by Theorem V, Art. 100, it appears that, If $f(h)$ is a function of h, which, as well as all its derived-functions up to the nth inclusively, is finite and continuous for all values of h between 0 and h; and if, in addition, $f(h)$ and its derived-functions up to the $(n-1)$th vanish when $h = 0$, then

$$f(h) = \frac{h^n}{1.2.3\ldots n} f^n(\theta h).$$

Accordingly
$$R_1 = \frac{h^2}{1.2} F''(x+\theta h), \tag{4}$$

and substituting this value in equation (2) above,

$$F(x+h) = F(x) + h F'(x) + \frac{h^2}{1.2} F''(x+\theta h), \tag{5}$$

which may be written in the form

$$F(x+h) = F(x) + h F'(x) + \frac{h^2}{1.2} F''(x) + R_2.$$

Whence $R_2 = F(x+h) - F(x) - h F'(x) - \frac{h^2}{1.2} F''(x)$; (6)

and therefore R_2 is a function of h, which vanishes when $h = 0$;

and its 1st derived $\frac{dR_2}{dh} = F'(x+h) - F'(x) - hF''(x) = 0$, when $h = 0$;

and its 2d derived $\frac{d^2R_2}{dh^2} = F''(x+h) - F''(x) = 0$, when $h = 0$;

and its 3d derived $\frac{d^3R_2}{dh^3} = F'''(x+h)$,

and which therefore does not vanish when $h = 0$. Hence, in accordance with the Theorem V, Art. 100, cited as above,

$$R_2 = \frac{h^3}{1.2.3} F'''(x+\theta h), \tag{7}$$

subject to the conditions that $F'(x)$, $F''(x)$, $F'''(x)$ are continuous and finite between the assigned limits. Substituting this value of R_2, we have

$$F(x+h) - F(x) - hF'(x) - \frac{h^2}{1.2} F''(x) = \frac{h^3}{1.2.3} F'''(x+\theta h);$$

and continuing in the same manner, if all the derived-functions of $F(x)$ are finite and continuous between the assigned limits up to the nth inclusively, we have

$$F(x+h) - F(x) - F'(x)\frac{h}{1} - \dots - \frac{h^{n-1}}{1.2.3\dots(n-1)} F^{n-1}(x)$$
$$= \frac{h^n}{1.2.3\dots n} F^n(x+\theta h);$$

and therefore

$$F(x+h) = F(x) + F'(x)\frac{h}{1} + F''(x)\frac{h^2}{1.2} + F'''(x)\frac{h^3}{1.2.3} + \dots$$
$$\dots + F^{n-1}(x)\frac{h^{n-1}}{1.2.3\dots(n-1)} + \frac{h^n}{1.2.3\dots n} F^n(x+\theta h). \quad (8)$$

This expression then gives the equivalent of $F(x+h)$ in terms of a series of ascending powers of h, and the conditions under which it has been formed shew in what cases the expansion is possible.

116.] In equation (8) the equality of the two members is perfect, and the development may be considered as completely effected, except so far as some indeterminateness arises from the nature of θ, to which quantity a specific value cannot be assigned. It was however before shewn that it must be some positive and proper fraction; and sometimes, if the series be what is called Convergent, when n is very great, the last terms and their sum become infinitesimal, and we must neglect

$$\frac{h^n}{1.2\dots n} F^n(x+\theta h),$$

and may write

$$F(x+h) = F(x) + F'(x)\frac{h}{1} + F''(x)\frac{h^2}{1.2} + \dots\dots \quad (9)$$

in which incomplete form the series was first given by Dr. Taylor, and now generally bears his name.

117.] In equation (8) let $x = 0$; that is, let us consider the function for all values of the variable between 0 and h; and let us write x for h, remembering that x is the superior limit; whereby the conditions are, that none of the functions or derived-functions are infinite or discontinuous for any value of x between 0 and x; then

$$F(x) = F(0) + F'(0)\frac{x}{1} + F''(0)\frac{x^2}{1.2} + \dots\dots$$

$$\dots\dots + F^{n-1}(0)\frac{x^{n-1}}{1.2.3\dots(n-1)} + \frac{x^n}{1.2.3\dots n}F^n(\theta x); \qquad (10)$$

this is Maclaurin's Theorem, of which an imperfect proof was given in Art. 54; and it accordingly appears that it is only a particular case of Taylor's. Many examples of both series having been given in Chapter III, it is unnecessary to add others. Of the general series (10) however the following are particular instances:

Let $n = 1$, then $F(x) = F(0) + x\,F'(\theta x)$. (11)

$$\text{Thus} \qquad e^x = 1 + xe^{\theta x}, \qquad \therefore \quad \frac{e^x - 1}{x} = e^{\theta x};$$

$$\sin x = x\cos\theta x, \qquad \frac{\sin x}{x} = \cos\theta x;$$

$$\log(1+x) = \frac{x}{1+\theta x}.$$

118.] If the series (10) be such that, as n increases $\frac{x^n}{1.2.3\dots n}F^n(\theta x)$ becomes infinitesimal, then

$$F(x) = F(0) + x\,F'(0) + \frac{x^2}{1.2}F''(0) + \frac{x^3}{1.2.3}F'''(0) + \dots\dots \qquad (12)$$

that is, the limit of the sum of the second member of the equation is $F(x)$.

119.] Again, in equation (8), Art. 115, for h write $a-x$, then

$$F(a) = F(x) + (a-x)\,F'(x) + \frac{(a-x)^2}{1.2}F''(x) + \dots\dots$$

$$+ \frac{(a-x)^{n-1}}{1.2.3\dots(n-1)}F^{n-1}(x) + \frac{(a-x)^n}{1.2.3\dots n}F^n\{x+\theta(a-x)\}. \qquad (13)$$

For x write a, and for a write x:

$$\text{F}(x) = \text{F}(a) + (x-a)\,\text{F}'(a) + \frac{(x-a)^2}{1.2}\,\text{F}''(a) + \ldots\ldots$$

$$+\frac{(x-a)^{n-1}}{1.2\ldots(n-1)}\,\text{F}^{n-1}(a) + \frac{(x-a)^n}{1.2\ldots(n-1)n}\,\text{F}^n\{a+\theta(x-a)\}, \quad (14)$$

the superior and inferior values of x in this case being respectively x and a; so that it is for all values of x between these limits that the conditions are to be satisfied.

As particular cases of the formula we have

$$\text{F}(x) = \text{F}(a) + (x-a)\,\text{F}'\{a+\theta(x-a)\}, \quad (15)$$

$$\text{F}(x) = \text{F}(a) + (x-a)\,\text{F}'(a) + \frac{(x-a)^2}{1.2}\,\text{F}''\{a+\theta(x-a)\}. \quad (16)$$

As an example of equation (14), let $\text{F}(x) = \log_e x$.

$$\therefore\quad \log x = \log a + \frac{x-a}{a} - \frac{1}{2}\left(\frac{x-a}{a}\right)^2 + \frac{1}{3}\left(\frac{x-a}{a}\right)^3 - \ldots\ldots$$

$$\ldots\ldots(-)^{n-2}\frac{1}{n-1}\left(\frac{x-a}{a}\right)^{n-1}(-)^{n-1}\frac{1}{n}\left(\frac{x-a}{a+\theta(x-a)}\right)^n \quad (17)$$

Other and important applications of this series will be given in future parts of the work.

120.] Hence it appears that if we stop at the nth term of Taylor's Series, and neglect to take account of all terms after it, the error committed by so doing is

$$\frac{h^n}{1.2.3\ldots n}\,\text{F}^n(x+\theta h); \quad (18)$$

and as θ lies between 0 and 1, the error lies between

$$\frac{h^n}{1.2.3\ldots n}\,\{\text{F}^n(x) \text{ and } \text{F}^n(x+h)\}.$$

Similarly, if we stop at the nth term of Maclaurin's Series, and do not take account of the terms after it, the error is

$$\frac{x^n}{1.2.3\ldots n}\,\text{F}^n(\theta x). \quad (19)$$

The expressions (18) and (19) are known by the names respectively of the *limits* of Taylor's and Maclaurin's Theorems. See Art. 55 and 67.

The series established in the last Article enables us to put the limit of Taylor's Series under another form which is sometimes convenient.

Let us represent the last term of (14) by $\phi(a)$; so that

$$\phi(a) = \frac{(x-a)^n}{1.2.3\ldots n} \mathrm{F}^n\{a+\theta(x-a)\}, \tag{20}$$

$$\therefore \quad \mathrm{F}(x) = \mathrm{F}(a) + \frac{x-a}{1}\mathrm{F}'(a) + \frac{(x-a)^2}{1.2}\mathrm{F}''(a) + \ldots\ldots$$

$$\ldots\ldots + \frac{(x-a)^{n-1}}{1.2.3\ldots(n-1)}\mathrm{F}^{n-1}(a) + \phi(a).$$

Differentiate this, making a the variable, and we have

$$\frac{(x-a)^{n-1}}{1.2.3\ldots(n-1)}\mathrm{F}^n(a) + \phi'(a) = 0. \tag{21}$$

It is plain from (20) that $\phi(x) = 0$; therefore, writing in (15) ϕ for F, we have

$$\phi(a) + (x-a)\,\phi'\{a+\theta_1(x-a)\} = 0,$$

θ_1 being different from θ in (20);

$$\therefore \quad \phi(a) = -(x-a)\,\phi'\{a+\theta_1(x-a)\}. \tag{22}$$

In (21) for a write $a + \theta_1(x-a)$, and we have

$$\frac{(x-a)^{n-1}(1-\theta_1)^{n-1}}{1.2.3\ldots(n-1)}\mathrm{F}^n\{a+\theta_1(x-a)\} + \phi'\{a+\theta_1(x-a)\} = 0; \tag{23}$$

whence, by elimination between (22) and (23),

$$\phi(a) = \frac{(1-\theta_1)^{n-1}(x-a)^n}{1.2.3\ldots(n-1)}\mathrm{F}^n\{a+\theta_1(x-a)\}; \tag{24}$$

and substituting in (14) this particular form of the remainder, we have

$$\mathrm{F}(x) = \mathrm{F}(a) + \frac{x-a}{1}\mathrm{F}'(a) + \frac{(x-a)^2}{1.2}\mathrm{F}''(a) + \ldots\ldots$$

$$+ \frac{(x-a)^{n-1}}{1.2\ldots(n-1)}\mathrm{F}^{n-1}(a) + \frac{(x-a)^n(1-\theta)^{n-1}}{1.2\ldots(n-1)}\mathrm{F}^n\{a+\theta(x-a)\}. \tag{25}$$

Substituting in this series $a+h$ for x, and subsequently writing x for a, we have

$$\text{F}(x+h) = \text{F}(x) + h\,\text{F}'(x) + \frac{h^2}{1.2}\,\text{F}''(x) + \ldots\ldots$$

$$\ldots + \frac{h^{n-1}}{1.2\ldots(n-1)}\,\text{F}^{n-1}(x) + \frac{h^n(1-\theta)^{n-1}}{1.2\ldots(n-1)}\,\text{F}^n(x+\theta h), \quad (26)$$

The corresponding expression for the remainder, in Maclaurin's Series, is

$$\frac{(1-\theta)^{n-1}x^n}{1.2.3\ldots(n-1)}\,\text{F}^n(\theta x). \quad (27)$$

Two examples of the remainders of series in these forms are subjoined.

Ex. 1. $\quad \text{F}(x) = a^x, \quad \therefore \quad \text{F}^n(x) = a^x(\log_e a)^n,$

and therefore the sum of all the terms after the nth in the expansion of a^x is

$$\frac{(1-\theta)^{n-1}x^n}{1.2.3\ldots(n-1)}(\log_e a)^n\,a^{\theta x}.$$

Ex. 2. $\quad \text{F}(x) = \sin mx, \quad \therefore \quad \text{F}^n(x) = m^n \sin\left(mx + n\frac{\pi}{2}\right);$

$\therefore$ the sum of all the terms after the nth of $\sin m(x+h)$ is

$$\frac{h^n(1-\theta)^{n-1}m^n}{1.2.3\ldots(n-1)}\sin\left\{m(x+\theta h) + n\frac{\pi}{2}\right\}.$$

121.] In these last Articles we have frequently used the expression "finite and continuous for all values of x between" certain limits; now although we cannot tell *all* the cases wherein these conditions are, and are not fulfilled, for a complete knowledge of *all* functions and *all* their properties would be necessary to our doing so, yet we do know certain cases where they are not satisfied; these we purpose to discuss briefly. In them Taylor's and Maclaurin's Series are said to *fail*, which is surely an incorrect term, for we are endeavouring to bring all functions within a particular formula, which is true only of *some;* that is, we are trying to make that which is true only when certain conditions are satisfied, comprise all, whether such conditions are fulfilled or not. We shall omit the consideration of discontinuous functions, as such are excluded by their very forms, and confine our attention to those which for particular values of the variable become infinite, and on that account fail to satisfy the requisite conditions.

First let us consider Taylor's Series. To develope $F(x+h)$ to $n+1$ terms, or so that the last term of the series may comprise the sum of all the terms after the nth, and be

$$F^n(x+\theta h)\frac{h^n}{1.2.3\ldots n},$$

it is necessary that $F(x)$, and all its derived-functions up to $F^n(x)$ inclusively, should *not be infinite for any value of* x *between* x *and* x + h. Now let the effects of differentiation on functions be observed; (1) that all rational and integral functions of x, and all functions of the form $x^n + cx^{-m} + ex^{\frac{p}{q}}$ are lowered one dimension by it, and that therefore, if n be positive and integral, the nth derived-function of x^n is a constant, and the $(n+1)$th is zero; that however many times x^{-m} be differentiated, the result is never zero, but that the negative exponent is increased; that if $\frac{p}{q}$ be a fraction proper or improper, some derived-function of it, sooner or later, has a negative exponent, and therefore some power of x appears in the denominator; (2) that if $e^{f(x)}$ be a factor of the original function, it is also a factor of every one of the successive derived-functions.

Excluding all consideration as to values outside those for which the series is applied, let a be a particular value of x within them; and suppose that when $x=a$, $F^n(x)$ is finite, but $F^{n+1}(x)$ is infinite: then it is plain that some factor of the form $x-a$ must, in passing from $F^n(x)$ to $F^{n+1}(x)$, have been introduced into the denominator; and therefore, as far as the above remarks go, there must have been in the original function a factor of the form $(x-a)^{m+\frac{p}{q}}$, where m is an integral and positive number, and $\frac{p}{q}$ is a *proper* fraction. And if the primitive and all its derived-functions are infinite, when $x=a$, there must in the original function have been a factor of the form $(x-a)^{-m}$.

Hence, in the latter case, the theorem of Taylor is not applicable at all for values which include $x=a$; and, in the former case, suppose we have to expand

$$F(x) = f(x) + (x-a)^{m+\frac{p}{q}}\,\phi(x),$$

in which neither $f(x)$ nor $\phi(x)$ involve factors of the form $x - a$, then all the derived-functions up to $\text{F}^m(x)$ will be finite, when $x = a$, but the subsequent ones will be infinite; the expansion therefore must not be carried beyond the mth term, but the addition of

$$\frac{h^m}{1.2\ldots m}\,\text{F}^m(x+\theta h),$$

will make the equation exact.

Suppose, for instance, that it is required to expand $\text{F}(a+h)$, having given

$$\text{F}(x) = x^4 + (x-a)^{\frac{5}{2}} \sin x;$$

then a is the least value of x for which the function is considered, and $a + h$ is the greatest; and

$$\text{F}'(x) = 4x^3 + \frac{5}{2}(x-a)^{\frac{3}{2}} \sin x + (x-a)^{\frac{5}{2}} \cos x,$$

$$\text{F}''(x) = 12x^2 + \frac{15}{4}(x-a)^{\frac{1}{2}} \sin x + 5(x-a)^{\frac{3}{2}} \cos x - (x-a)^{\frac{5}{2}} \sin x.$$

But if we form $\text{F}'''(x)$, it involves $(x-a)^{-\frac{1}{2}}$, which becomes infinite when $x = a$, and therefore fails to fulfil the conditions under which equation (8) has been determined: therefore, in this case,

$$\text{F}(x+h) = \text{F}(x) + \frac{h}{1}\text{F}'(x) + \frac{h^2}{1.2}\text{F}''(x+\theta h);$$

and substituting the specific values above given, and putting $x = a$,

$$\text{F}(a+h) = a^4 + 4a^3\frac{h}{1} + \frac{h^2}{1.2}\{12(a+\theta h)^2$$

$$+ \frac{15}{4}(\theta h)^{\frac{1}{2}} \sin(a+\theta h) + 5(\theta h)^{\frac{3}{2}} \cos(a+\theta h) - (\theta h)^{\frac{5}{2}} \sin(a+\theta h)\},$$

θ being a positive and proper fraction. On inspection it is plain that the first three terms of the series are correct terms of the expansion of $(a+h)^4$, and that they might have been carried further; and that the last three express $(x-a)^{\frac{5}{2}} \sin x$, when $x = a + h$.

Now observing that the three last terms involve $h^{\frac{1}{2}}$, we have a good illustration of the reason why Taylor's Series "fails;" viz. because we are attempting to expand by a formula which

involves only integral powers of h, a function which in its development requires fractional powers. Another and perhaps better illustration is the following: Let us consider the case of $(x^2-a^2)^{\frac{1}{2}}$, which is to be developed, when $x = a+h$, so that the original function $= 0$ when $x = a$, and the first-derived is ∞; but as the function may be written in the form $(x-a)^{\frac{1}{2}}(x+a)^{\frac{1}{2}}$, this becomes, when $x = a+h$, $h^{\frac{1}{2}}(2a+h)^{\frac{1}{2}}$, the second factor of which may be developed in the ordinary way, and thereby the whole development will consist of terms of the powers of h whose exponents are of the form $n+\frac{1}{2}$, n being a whole number.

Similarly, if the original function has factors of the form $(x-a)^{-m}$, that and all its derived-functions are infinite when $x = a$. Thus suppose we have to develope by Taylor's Series, when $x = a+h$,

$$\text{F}(x) = \frac{\sin x}{x^2-a^2},$$

this and all its derived-functions become infinite when $x = a$; but the function may be written in the form

$$\frac{1}{x-a}\,\frac{\sin x}{x+a},$$

the first factor of which, when $x = a+h$, becomes $\frac{1}{h}$, and the second fulfils the conditions of Taylor's Series and may be expanded in the ordinary way; but the resulting development will have at least one term involving a negative power of h, which indicates the cause of the failure, viz. that the function does not admit of development in ascending integral and positive powers of h, which alone is given by Taylor's Theorem.

Secondly, as to the failure of Maclaurin's Theorem. There are two cases corresponding to those above discussed; one when the original function involves factors of the forms x^{-m} and $x^{m+\frac{p}{q}}$, the former of which is always infinite when $x = 0$, that is, at the lower value of x, and the latter of which, when differentiated more than m times, will have negative indices, and thereby will be infinite when $x = 0$. And another case is $\text{F}(x) = \log x$, which $= \infty$ when $x = 0$, and so do all its successive derived-functions.

Also, if the function to be expanded by Maclaurin's Theorem be

$$F(x) = f(x) + e^{-\frac{1}{x^2}},$$

as $e^{-\frac{1}{x^2}}$ and all its derived-functions vanish when $x = 0$, the development of $F(x)$ will be that of $f(x)$ alone, and therefore will be inexact for $f(x) + e^{-\frac{1}{x^2}}$; the reason of the inexactness being that $e^{-\frac{1}{x^2}}$ does not admit of development into a series of ascending integral and positive powers of x, which alone Maclaurin's Theorem gives.

There may also be many other functions which are not capable of development in the form of Taylor's Series, and therefore the student must be on his guard against attempts at development in a form not suited to the function; and the above remarks must be considered as general hints rather than as a secure and scientific enumeration of cases of applicability and non-applicability; the latter is beyond our knowledge, and therefore with the former, however incomplete, we must be content.

We have treated thus at length of the cases where Taylor's and Maclaurin's Theorems fail, because some foreign and most English writers have attempted to raise the Differential Calculus on them as its basis. As far as they are applicable, reasoning founded on them may be correct; but since they are not universally so, it is objected, and validly objected, that the basis of the Calculus is confined within limits narrower than is necessary. And no criteria are known for determining whether functions can be expanded in *their* forms or not, before such principles of continuity, as those which we have made fundamental, have been elucidated; on the expansion-principles therefore we are left to grope our way in the dark, being uncertain whether the matter which we are discussing is within their comprehension or not.

SECTION 2.—*On Functions of two or more variables.*

122.] Let $F(x, y)$ be a function of x, y; our object is to find the value of $F(x+h, y+k)$, in ascending powers of h and k, $F(x, y)$ and all its derived-functions being finite and continuous for all values of the variables between x, y and $x+h$, $y+k$, h and k being finite increments of x and y.

Let us consider the finite increments of x and y to be ht and kt, so that ultimately they may be reduced to h and k by putting $t = 1$; our object is to expand $F(x+ht, \ y+kt)$, which we will consider to be a function of t, so that

$$f(t) = F(x+ht, y+kt), \tag{28}$$

and therefore
$$f(0) = F(x, y). \tag{29}$$

Then, by series (10), Art. 117,

$$f(t) = f(0) + tf'(0) + \frac{t^2}{1.2}f''(0) + \dots + \frac{t^{n-1}}{1.2\dots(n-1)}f^{n-1}(0)$$
$$+ \frac{t^n}{1.2.3\dots n}f^n(t\theta). \tag{30}$$

To calculate the several derived-functions of $f(t)$; let

$$\left.\begin{aligned} x+ht &= x', & \therefore \quad \frac{dx'}{dt} &= h; \\ y+kt &= y', & \frac{dy'}{dt} &= k; \end{aligned}\right\} \tag{31}$$

$$\therefore \quad f(t) = F(x', y');$$

$$\therefore \quad f'(t) = \frac{d.f(t)}{dt} = \left(\frac{dF}{dx'}\right)\frac{dx'}{dt} + \left(\frac{dF}{dy'}\right)\frac{dy'}{dt},$$

$$= \left(\frac{dF}{dx'}\right) h + \left(\frac{dF}{dy'}\right) k;$$

and observing, by (31), that, when $t = 0$, $x' = x$ and $y' = y$, we have

$$f'(0) = \left(\frac{dF}{dx}\right) h + \left(\frac{dF}{dy}\right) k. \tag{32}$$

Similarly,

$$f''(t) = \frac{df'(t)}{dt} = \left(\frac{d^2F}{dx'^2}\right)\frac{dx'^2}{dt^2}$$
$$+ 2\left(\frac{d^2F}{dx'dy'}\right)\frac{dx'dy'}{dt^2} + \left(\frac{d^2F}{dy'^2}\right)\frac{dy'^2}{dt^2}, \tag{33}$$

remembering that t is equicrescent, otherwise equation (30) would not be true;

$$\therefore \quad f''(0) = \left(\frac{d^2F}{dx^2}\right) h^2 + 2\left(\frac{d^2F}{dxdy}\right) hk + \left(\frac{d^2F}{dy^2}\right) k^2. \tag{34}$$

Similarly,

$$f'''(0) = \left(\frac{d^3\text{F}}{dx^3}\right) h^3 + 3\left(\frac{d^3\text{F}}{dx^2\,dy}\right) h^2k + 3\left(\frac{d^3\text{F}}{dx\,dy^2}\right) hk^2 + \left(\frac{d^3\text{F}}{dy^3}\right) k^3$$

. . . .

$$f^n(0) = \left(\frac{d^n\text{F}}{dx^n}\right) h^n + n\left(\frac{d^n\text{F}}{dx^{n-1}\,dy}\right) h^{n-1}k$$
$$+ \frac{n(n-1)}{1.2}\left(\frac{d^n\text{F}}{dx^{n-2}\,dy^2}\right) h^{n-2}k^2 + \ldots\ldots \quad (35)$$

And substituting these values in the equation above, we have

$$\text{F}(x + ht, y + kt) = \text{F}(x, y) + \frac{t}{1}\left\{\left(\frac{d\text{F}}{dx}\right) h + \left(\frac{d\text{F}}{dy}\right) k\right\}$$
$$+ \frac{t^2}{1.2}\left\{\left(\frac{d^2\text{F}}{dx^2}\right) h^2 + 2\left(\frac{d^2\text{F}}{dx\,dy}\right) hk + \left(\frac{d^2\text{F}}{dy^2}\right) k^2\right\}$$
$$+ \frac{t^3}{1.2.3}\left\{\left(\frac{d^3\text{F}}{dx^3}\right) h^3 + 3\left(\frac{d^3\text{F}}{dx^2\,dy}\right) h^2k + \ldots\ldots\right\} + \ldots\ldots$$
$$+ \frac{t^n}{1.2.3\ldots n}\left\{\left(\frac{d^n\text{F}}{dx^n}\right) h^n + n\left(\frac{d^n\text{F}}{dx^{n-1}\,dy}\right) h^{n-1}k + \ldots\ldots\right.$$
$$\left.\ldots + \left(\frac{d^n\text{F}}{dy^n}\right) k^n\right\} \begin{Bmatrix} x + \theta ht, \\ y + \theta kt, \end{Bmatrix} \quad (36)$$

the meaning of the notation in the last line being, that x and y are to be replaced by $x + \theta ht$, $y + \theta kt$, in the coefficient of t^n; let $t = 1$, whence

$$\text{F}(x + h, y + k) = \text{F}(x, y) + \left(\frac{d\text{F}}{dx}\right) h + \left(\frac{d\text{F}}{dy}\right) k$$
$$+ \frac{1}{1.2}\left\{\left(\frac{d^2\text{F}}{dx^2}\right) h^2 + 2\left(\frac{d^2\text{F}}{dx\,dy}\right) hk + \left(\frac{d^2\text{F}}{dy^2}\right) k^2\right\}$$
$$+ \frac{1}{1.2.3}\left\{\left(\frac{d^3\text{F}}{dx^3}\right) h^3 + 3\left(\frac{d^3\text{F}}{dx^2\,dy}\right) h^2k + \ldots\ldots\right\}$$
$$+ \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ .$$
$$+ \frac{1}{1.2.3\ldots n}\left\{\left(\frac{d^n\text{F}}{dx^n}\right) h^n + n\left(\frac{d^n\text{F}}{dx^{n-1}\,dy}\right) h^{n-1}k + \ldots\ldots\right.$$
$$\left.\ldots + \left(\frac{d^n\text{F}}{dy^n}\right) k^n\right\} \begin{Bmatrix} x + \theta h, \\ y + \theta k. \end{Bmatrix} \quad (37)$$

Ex. 1. Given that $F(x, y) = x^2(a+y)^3$, it is required to find $(x + h)^2 (a + y + k)^3$.

$$\left(\frac{dF}{dx}\right) = 2x(a+y)^3, \quad \left(\frac{d^2F}{dx^2}\right) = 2(a+y)^3, \quad \left(\frac{d^3F}{dx^3}\right) = 0,$$

$$\left(\frac{dF}{dy}\right) = 3x^2(a+y)^2, \quad \left(\frac{d^2F}{dx\,dy}\right) = 6x(a+y)^2, \quad \left(\frac{d^3F}{dx^2\,dy}\right) = 6(a+y)^2,$$

$$\left(\frac{d^2F}{dy^2}\right) = 6x^2(a+y), \quad \left(\frac{d^3F}{dx\,dy^2}\right) = 12x(a+y),$$

$$\left(\frac{d^3F}{dy^3}\right) = 6x^2.$$

$$\left(\frac{d^4F}{dx^4}\right) = 0,$$

$$\left(\frac{d^4F}{dx^3\,dy}\right) = 0,$$

$$\left(\frac{d^4F}{dx^2\,dy^2}\right) = 12(a+y),$$

$$\left(\frac{d^4F}{dx\,dy^3}\right) = 12x, \quad \left(\frac{d^5F}{dx^2\,dy^3}\right) = 12,$$

$$\left(\frac{d^4F}{dy^4}\right) = 0;$$

whence, substituting in equation (37)

$$(x + h)^2 (a + y + k)^3 = x^2 (a + y)^3$$

$$+ \frac{1}{1} \left\{2x(a+y)^3 h + 3x^2(a+y)^2 k\right\}$$

$$+ \frac{1}{1.2} \left\{2(a+y)^3 h^2 + 12x(a+y)^2 hk + 6x^2(a+y) k^2\right\}$$

$$+ \frac{1}{1.2.3} \left\{18(a+y)^2 h^2 k + 36x(a+y)\, hk^2 + 6x^2 k^3\right\}$$

$$+ \frac{1}{1.2.3.4} \left\{72(a+y)\, h^2 k^2 + 48\, xhk^3\right\}$$

$$+ \frac{1}{1.2.3.4.5} \quad 120h^2k^3.$$

123.] If it be required to expand $F(x+h, y+k, z+l\ldots\ldots)$, then, by a similar process, we have

$$
\begin{aligned}
F(x+h, y+k, z+l\ldots\ldots) &= F(x, y, z\ldots\ldots) \\
&+ \left\{\left(\frac{dF}{dx}\right)h + \left(\frac{dF}{dy}\right)k + \left(\frac{dF}{dz}\right)l + \ldots\ldots\right\} \\
&+ \frac{1}{1.2}\left\{\left(\frac{d^2F}{dx^2}\right)h^2 + \left(\frac{d^2F}{dy^2}\right)k^2 + \ldots\ldots \right. \\
&\qquad \left. + 2\left(\frac{d^2F}{dx\,dy}\right)hk + 2\left(\frac{d^2F}{dx\,dz}\right)hl + \ldots\ldots\right\} \\
&+ \frac{1}{1.2.3}\left\{\left(\frac{d^3F}{dx^3}\right)h^3 + \ldots + 3\left(\frac{d^3F}{dx^2\,dy}\right)h^2k + \ldots\ldots \right. \\
&\qquad \left. \ldots + 3\left(\frac{d^3F}{dx\,dy^2}\right)hk^2 + \ldots\ldots\right\} \\
&+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&+ \frac{1}{1.2.3\ldots n}\left\{\left(\frac{d^nF}{dx^n}\right)h^n + \left(\frac{d^nF}{dy^n}\right)k^n + \ldots + n\left(\frac{d^nF}{dx^{n-1}\,dy}\right)h^{n-1}k \right. \\
&\qquad \left. + \ldots\ldots\right\}\left\{\begin{bmatrix} x+\theta h, \\ y+\theta k, \\ z+\theta l, \\ \ldots \\ \ldots \end{bmatrix}\right. \qquad (38)
\end{aligned}
$$

replacing $x, y, z\ldots$ in the last term by $x+\theta h$, $y+\theta k$, $z+\theta l, \ldots$

As the equations (37) and (38) stand at present, each side is exactly equal to the other; but if we can assure ourselves that as n increases without limit, each term of the remainder decreases without limit, then the remainders may be neglected, and the equations will be modified accordingly.

Ex. 1. If $F(x, y, z) = Ax^2 + By^2 + Cz^2 + 2Eyz + 2Gzx + 2Hxy + K$; it is required to expand $F(x+h, y+k, z+l)$.

$$\left(\frac{dF}{dx}\right) = 2Ax + 2Gz + 2Hy, \qquad \left(\frac{d^2F}{dx^2}\right) = 2A, \qquad \left(\frac{d^2F}{dy\,dz}\right) = 2E,$$

$$\left(\frac{dF}{dy}\right) = 2By + 2Ez + 2Hx, \qquad \left(\frac{d^2F}{dy^2}\right) = 2B, \qquad \left(\frac{d^2F}{dz\,dx}\right) = 2G,$$

$$\left(\frac{dF}{dz}\right) = 2Cz + 2Ey + 2Gx, \qquad \left(\frac{d^2F}{dz^2}\right) = 2C, \qquad \left(\frac{d^2F}{dx\,dy}\right) = 2H.$$

$$\therefore\ \mathrm{F}(x+h, y+k, z+l) = \mathrm{A}x^2+\mathrm{B}y^2+\mathrm{C}z^2+2\mathrm{E}yz+2\mathrm{G}zx+2\mathrm{H}xy+\mathrm{K}$$

$$+\frac{1}{1}\left\{(2\mathrm{A}x+2\mathrm{G}z+2\mathrm{H}y)h + (2\mathrm{B}y+2\mathrm{E}z+2\mathrm{H}x)k \right.$$
$$\left. + (2\mathrm{C}z+2\mathrm{E}y+2\mathrm{G}x)l\right\}$$

$$+\frac{1}{1.2}\left\{2\mathrm{A}h^2+2\mathrm{B}k^2+2\mathrm{C}l^2+4\mathrm{E}kl+4\mathrm{G}lh+4\mathrm{H}hk\right\}.$$

124.] If in the preceding formula (37) we make $x = 0$, $y = 0$, and then change h and k into x and y, we have

$$\mathrm{F}(x,y) = \mathrm{F}(0,0) + \left(\frac{d\mathrm{F}}{dx}\right)_0 x + \left(\frac{d\mathrm{F}}{dy}\right)_0 y,$$
$$+\frac{1}{1.2}\left\{\left(\frac{d^2\mathrm{F}}{dx^2}\right)_0 x^2 + 2\left(\frac{d^2\mathrm{F}}{dx\,dy}\right)_0 xy + \left(\frac{d^2\mathrm{F}}{dy^2}\right)_0 y^2\right\}$$
$$+ \quad \cdots\cdots\cdots\cdots$$
$$+\frac{1}{1.2\ldots n}\left\{\left(\frac{d^n\mathrm{F}}{dx^n}\right) x^n + n\left(\frac{d^n\mathrm{F}}{dx^{n-1}\,dy}\right) x^{n-1}y + \ldots + \left(\frac{d^n\mathrm{F}}{dy^n}\right) y^n\right\}\begin{Bmatrix}\theta x,\\ \theta y,\end{Bmatrix}$$

where we have to replace x and y by the value 0 in all the partial derived-functions, except in those of the last term, where they are to be replaced by θx and θy; and if this last term decreases without limit, as n increases without limit, then the remainder may be neglected, and the series may be written without the last term. Similarly may functions of more than two variables be expanded in ascending powers of the variables.

CHAPTER VII.

ON THE DETERMINATION OF MAXIMA AND MINIMA VALUES OF FUNCTIONS.

125.] CONSIDER a function of a single variable x; and, to fix our thoughts, let the variable continuously increase, then the corresponding variation of the function need not always be one of increase or of decrease, but it may increase up to a certain value and afterwards decrease, or *vice versâ*. In the former of these two cases, at the value of the variable when the function ceases to increase, it has attained a *greatest* value, or what is technically called a *maximum* state; and in the latter it reaches a least or *minimum* state; such *singular* conditions of a function the principles of Chapter IV enable us to determine. And we define as follows:

DEF.—A particular value of a function, which is greater than *all* its values in the immediate neighbourhood, that is, when the variables are infinitesimally increased or decreased, is said to be *a maximum*. And the particular value which is less than all its immediately adjacent ones, is called *a minimum*.

Maxima and *minima* are therefore terms not used absolutely, but in reference to the values of the functions immediately adjacent to those to which the names are applied.

As a simple illustration, consider $\sin x$; and let the radius of the circle be unity, then, when the arc $= 0$, the sine $= 0$; but as the arc increases up to $\frac{\pi}{2}$, the sine increases and at last becomes 1, which is its maximum, for as the arc becomes larger the sine becomes smaller, and continually decreases, passing through 0 when the arc $= \pi$, until, when the arc $= \frac{3\pi}{2}$, the sine $= -1$; after which it continually increases until, when the arc $= \frac{5\pi}{2}$, the sine $= +1$, which is a maximum, and so on. Thus as the arc increases, the sine periodically attains to maxima and minima values.

SECTION 1.—*On Maxima and Minima of Explicit Functions of One Variable.*

126.] Let $y = \text{F}(x)$ be the function of which the maxima and minima are to be investigated.

From the definition it is plain that if, as x increases up to a certain value x_0, $\text{F}(x)$ increases, and afterwards as x increases, $\text{F}(x)$ decreases, then $\text{F}(x)$ has attained a maximum value at $x = x_0$; and if, as x increases up to a certain value x_0, $\text{F}(x)$ decreases and afterwards increases, then $\text{F}(x_0)$ is a minimum value.

Now Theorem I, Chapter IV, is immediately applicable to determine these conditions: if x and $\text{F}(x)$ are simultaneously increasing, $\text{F}'(x)$ is positive; but if as x increases, $\text{F}(x)$ decreases, $\text{F}'(x)$ is negative.

If therefore at any point $x = x_0$, $\text{F}'(x)$ changes its sign from $+$ to $-$, we have a maximum value; and if $\text{F}'(x)$ changes its sign from $-$ to $+$, we have a minimum value; and as changes of sign can take place only when the quantity passes through 0 or ∞, we have the following rule to determine Maxima and Minima:

RULE.—Find every value of x which renders $\text{F}'(x)$ equal to 0 and to ∞; if such a value makes $\text{F}'(x)$ change its sign, the corresponding value of $\text{F}(x)$ is a maximum or minimum: being a maximum if $\text{F}'(x)$ changes sign from $+$ to $-$, and a minimum if the change be from $-$ to $+$; but if there is no change of sign, there is no such singular value.

Ex. 1. $$y = \text{F}(x) = x^2-2ax,$$

$$\therefore \; \text{F}'(x) = 2x-2a = 0, \text{ if } x = a;$$

and as $\text{F}'(x)$ is negative when x is less than a, and positive when x is greater than a, $\text{F}'(x)$ changes sign from $-$ to $+$ in passing through a, and therefore $\text{F}(x)$ has a minimum value, viz. $-a^2$.

Ex. 2. $$y = \text{F}(x) = \sin x,$$

$$\text{F}'(x) = \cos x = 0, \text{ if } x = \frac{\pi}{2},$$

and as $\cos x$ is positive when x is less than $\frac{\pi}{2}$, and negative when x is greater than $\frac{\pi}{2}$, $\text{F}'(x)$ changes sign from $+$ to $-$, and accordingly $\text{F}(x)$ has a maximum value, viz. 1.

Also since $\text{F}'(x) = \cos x = 0$, when $x = \frac{3\pi}{2}$, and changes sign from $-$ to $+$ as x passes through this value, $\sin x$ has at it a minimum value, viz. -1. Similarly also $x = \frac{5\pi}{2}$ gives $\cos x$ a change of sign from $+$ to $-$, and therefore gives to $\sin x$ a maximum, viz. 1; and thus may other values be determined.

Ex. 3. $$y = x(a-x)^2 = \text{F}(x),$$

$$\therefore \; \text{F}'(x) = (a-x)(a-3x) = 0, \text{ if}$$

$x = a$, and changes sign from $-$ to $+$, $\therefore$ a minimum;

$x = \frac{a}{3}$, ... from $+$ to $-$, $\therefore$ a maximum;

$\therefore$ 0 is a minimum value of $x(a-x)^2$, viz. when $x = a$; and when $x = \frac{a}{3}$, $\text{F}(x) = \frac{4a^3}{27}$, which is a maximum value.

It is convenient to have a distinctive name for that value of a variable which makes a function of it to vanish, and therefore we propose to call it the *critical* value; thus a and $\frac{a}{3}$ are critical values of x in $\text{F}'(x)$ in the last example; 0 and a would be critical values of $x(a-x)^2$. It is plain that critical values do not necessarily cause a function to change sign, although a function cannot change sign except at a critical value: at least such is the case as far as our knowledge goes.

127.] When, as in Ex. 3 above, $\text{F}'(x)$ is an algebraical function, and has many factors which, when equated to 0, cause it to vanish, it is easy to perceive what must be their forms that $\text{F}'(x)$ may change its sign. Corresponding to every factor of uneven dimensions, that is, of the form $(x-x_0)^{2m+1}$, as x passes through x_0, there is a change of sign; but to factors of even dimensions, viz. of the form $(x-x_0)^{2m}$, there is no change of sign as x passes through x_0, and therefore there is no maximum or minimum value.

To determine the change of sign corresponding to a factor of uneven dimensions, the best method is first to determine the signs of all the factors, short of the critical factor, corresponding to the critical value, and then to investigate the change of sign of the critical factor; the following example will explain the process: Suppose

$$F'(x) = x^3 (x-1)^2 (x-2)^3 (x-4)^4,$$

which is equal to 0, if

$x = 0$, and gives a change of sign from + to −, ∴ a maximum;
$x = 1$, and gives no change of sign, ∴ no max. or minimum;
$x = 2$, and gives a change of sign from − to +, ∴ a minimum;
$x = 4$, and gives no change of sign, ∴ no max. or minimum.

that is, if $x = 0$, the critical factor is x^3; but when $x = 0$, the other factors severally are $+1$, -8, $+16$, the product of which is −; and as x^3, when $x = 0$, changes sign from − to +, it follows that $F'(x)$, when $x = 0$, changes sign from + to −, and accordingly $F(x)$ has a corresponding maximum value; by a similar method must the changes of sign due to the other critical factors be determined.

128.] Geometrical illustrations of the several conditions of maxima and minima are given in figs. 12, 13, 14, 15.

Suppose $y = F(x)$ to represent a curve such as those drawn in the figs.

Let $OM_0 = x_0$, $M_0P_0 = y_0$ the corresponding ordinate.

Then in fig. 12, as x increases up to x_0, $y = F(x)$ increases, and therefore $F'(x)$ is positive; but as soon as x passes the value x_0, y begins to decrease, and $F'(x)$ is negative, and the ordinate y or $F(x)$ has manifestly attained a maximum value at x_0.

In fig. 13, the reverse is the case; as x increases up to x_0 $y = F(x)$ decreases, but as soon as x is greater than x_0, $F(x)$ increases, and thus the sign of $F'(x)$ changes from − to + at x_0, and the corresponding value of $F(x)$ is a minimum.

Fig. 14 illustrates the case of $F'(x)$ being positive up to x_0, and although $F'(x) = 0$, yet it does not change its sign, but continues positive afterwards, and therefore we have no maximum value.

In the curve drawn in fig. 15, $F'(x)$ is negative throughout; at P_0 it is equal to 0, but as it does not change its sign, there is no minimum value.

129.] **Examples of maxima and minima.**

Ex. 1. To determine the maxima and minima values of y, having given

$$y = F(x) = x^4-8ax^3+22a^2x^2-24a^3x+12a^4,$$

$$\frac{dy}{dx} = F'(x) = 4x^3-24ax^2+44a^2x-24a^3,$$

$$= 4(x-a)(x-2a)(x-3a) = 0,$$

if $x = a$, and changes sign from $-$ to $+$, $\therefore$ a minimum;

$x = 2a$, ... $+$ to $-$, $\therefore$ a maximum;

$x = 3a$, ... $-$ to $+$, $\therefore$ a minimum;

whence we have, if $x = a$, $F(x) = 3a^4$, a minimum;

$x = 2a$, $F(x) = 4a^4$, a maximum;

$x = 3a$, $F(x) = 3a^4$, a minimum.

Ex. 2. To determine the maxima and minima values of

$$F(x) = (x-1)^4(x+2)^3,$$

$$F'(x) = (x-1)^3(x+2)^2(7x+5) = 0, \text{ if}$$

$x = 1$, and changes sign from $-$ to $+$, $\therefore$ a minimum;

$x = -2$, but does not change sign, $\therefore$ no max. or minimum;

$x = -\frac{5}{7}$, and changes sign from $+$ to $-$, $\therefore$ a maximum.

Hence, when $x = 1$, $F(x) = 0$, a minimum value.

. . . . . . $x = -\frac{5}{7}$, $F(x) = \frac{12^4\,9^3}{7^7}$, a maximum.

Ex. 3. To determine the maximum and minimum values of $F(x)$, having given

$$F(x) = \frac{(x+2)^3}{(x-3)^2},$$

$$\therefore \quad F'(x) = \frac{(x+2)^2(x-13)}{(x-3)^3};$$

$= 0$, if $x = -2$, and does not change sign, $\therefore$ no max. or min.;

$= 0$, if $x = 13$, and changes sign from $-$ to $+$, $\therefore$ a minimum;

$= \infty$, if $x = 3$, ... $+$ to $-$, $\therefore$ a maximum;

Hence, when $x = 13$, $F(x) = \frac{135}{4}$, a minimum value;

. . . . . . $x = 3$, $F(x) = \infty$, a maximum value.

Ex. 4. To determine the minimum value of x^x.

$$\text{F}(x) = x^x,$$

$$\text{F}'(x) = x^x(1+\log x) = 0,$$

if $\log x = -1$, that is, if

$$x = \frac{1}{e};$$

and as $\text{F}'(x)$ changes sign from $-$ to $+$, the corresponding value of x^x, viz.

$$\left(\frac{1}{e}\right)^{\frac{1}{e}}$$

is a minimum.

Ex. 5. Required to find the value of x, when $\sin x + \cos x$ is a maximum.

$$\text{F}(x) = \sin x + \cos x,$$

$$\text{F}'(x) = \cos x - \sin x = 0,$$

if $\tan x = 1$; that is, when

$x = \frac{\pi}{4}$, and changes sign from $+$ to $-$, $\therefore$ a maximum;

$x = \frac{5\pi}{4}$, ... $-$ to $+$, $\therefore$ a minimum.

. . . .

Hence the maximum value of the function is $\sqrt{2}$, and the minimum is $-\sqrt{2}$; which values recur whenever x is increased by π.

130.] The change of sign of $\text{F}'(x)$ may often be conveniently determined from the following considerations.

Suppose that $\text{F}''(x)$ does not vanish or become infinite when $\text{F}'(x) = 0$; then since $\text{F}''(x) = \dfrac{d^2y}{dx^2} = \dfrac{d.\left(\frac{dy}{dx}\right)}{dx} = \dfrac{d.\text{F}'(x)}{dx}$, it is manifest that if x increases, $\text{F}''(x)$ is positive or negative according as $\text{F}'(x)$ simultaneously increases or decreases; but if as x increases, $\text{F}'(x)$ changes sign from $-$ to $+$, it is increasing, and if it changes sign from $+$ to $-$, it is decreasing; hence

for a minimum value $F''(x)$ is positive, and for a maximum value $F''(x)$ is negative. Accordingly

$F(x)$ is a maximum or minimum value, according as the value of x, determined from the equation $F'(x) = 0$, renders $F''(x)$ negative or positive.

Ex. 1. Let $F(x) = \dfrac{x}{1+x^2}$,

$$F'(x) = \frac{1-x^2}{(1+x^2)^2} = 0, \text{ if } x = +1, \text{ and if } x = -1,$$

$$F''(x) = \frac{2x^3-6x}{(1+x^2)^3};$$

and if $x = +1$, $F''(x) = -\dfrac{1}{2}$, $\therefore$ $F(x) = \dfrac{1}{2}$, a maximum;

. . . $x = -1$, $F''(x) = +\dfrac{1}{2}$, $\therefore$ $F(x) = -\dfrac{1}{2}$, a minimum.

Ex. 2.
$$\begin{aligned} F(x) &= \sin x \cos (a-x), \\ F'(x) &= \cos x \cos (a-x) + \sin x \sin (a-x), \\ &= \cos (a-2x), \\ F''(x) &= 2 \sin (a-2x); \end{aligned}$$

$$\therefore \quad F'(x) = 0, \text{ if } a-2x = -\frac{\pi}{2}, \quad \therefore \ x = \frac{a}{2}+\frac{\pi}{4};$$

$$. \ . \ . = 0, \text{ if } a-2x = \frac{\pi}{2}, \quad \therefore \ x = \frac{a}{2}-\frac{\pi}{4}.$$

In the former of which cases $F''(x)$ is negative, and the corresponding value of $F(x)$ is a maximum; and in the latter $F''(x)$ is positive, and the corresponding value of $F(x)$ is a minimum.

131.] This method however of determining such singular values of functions is not applicable whenever the value of x which makes $F'(x) = 0$, also makes $F''(x) = 0$; in which case, as in all others, the following method may be employed.

Let $F(x)$ be the function of which the maximum and minimum values are to be determined; then, by equation (21), Art. 101,

$$F(x+h) - F(x) = hF'(x+\theta h).$$

As h becomes infinitesimal $F'(x+\theta h)$ approaches to its limiting value $F'(x)$.

Suppose now that x_0 is such a value of x that $\mathrm{F}(x_0)$ is a maximum or minimum; then, if h diminishes without limit and $\mathrm{F}'(x_0)$ does not vanish, we have

$$\mathrm{F}(x_0+h)-\mathrm{F}(x_0)=h\mathrm{F}'(x_0);$$

but when this is the case, the second member of the equation changes sign with h; that is, $\mathrm{F}(x_0+h)-\mathrm{F}(x_0)$ and $\mathrm{F}(x_0-h)-\mathrm{F}(x_0)$ are not of the same sign; which is inconsistent with $\mathrm{F}(x_0)$ being a maximum or a minimum.

Suppose however that $\mathrm{F}'(x_0)=0$, and that $\mathrm{F}''(x_0)$ does not vanish, then, by equation (22), Art. 101,

$$\mathrm{F}(x_0+h)-\mathrm{F}(x_0)=\frac{h^2}{1.2}\mathrm{F}''(x_0+\theta h);$$

which becomes, when h diminishes without limit,

$$\mathrm{F}(x_0+h)-\mathrm{F}(x_0)=\frac{h^2}{1.2}\mathrm{F}''(x_0).$$

If therefore $\mathrm{F}''(x_0)$ is positive, $\mathrm{F}(x_0)$ is less than both $\mathrm{F}(x_0+h)$ and $\mathrm{F}(x_0-h)$; and if $\mathrm{F}''(x_0)$ is negative, $\mathrm{F}(x_0)$ is greater than both $\mathrm{F}(x_0+h)$ and $\mathrm{F}(x_0-h)$; whence we conclude,

If $\mathrm{F}'(x_0)=0$, and $\mathrm{F}''(x_0)$ does not vanish; if $\mathrm{F}''(x_0)$ is negative, $\mathrm{F}(x_0)$ is a maximum, and if $\mathrm{F}''(x_0)$ is positive, $\mathrm{F}(x_0)$ is a minimum.

But again, if $\mathrm{F}''(x_0)=0$, and $\mathrm{F}'''(x_0)$ does not vanish,

$$\mathrm{F}(x_0+h)-\mathrm{F}(x_0)=\frac{h^3}{1.2.3}\mathrm{F}'''(x_0+\theta h);$$

in which case, as h^3 changes its sign with h, it is plain that there is no maximum or minimum value; but if $\mathrm{F}'''(x_0)=0$, and $\mathrm{F}''''(x_0)$ does not vanish, then

$$\mathrm{F}(x_0+h)-\mathrm{F}(x_0)=\frac{h^4}{1.2.3.4}\mathrm{F}''''(x_0+\theta h);$$

in which case, as before, there will be a maximum or minimum value of $\mathrm{F}(x)$, according as $\mathrm{F}''''(x_0)$ is negative or positive.

And thus generally if the value x_0, which makes $\mathrm{F}'(x)=0$, so affects $\mathrm{F}''(x)$, $\mathrm{F}'''(x)$, ... up to $\mathrm{F}^{n-1}(x)$, that all vanish, but that $\mathrm{F}^n(x_0)$ does not vanish, then we have

$$\mathrm{F}(x_0+h)-\mathrm{F}(x_0)=\frac{h^n}{1.2.3\ldots n}\mathrm{F}^n(x_0+\theta h),$$

and if n be an odd number, there is no maximum or minimum value; but if n be an even number, $\mathrm{F}(x_0)$ is a maximum if $\mathrm{F}^n(x_0)$ is negative, and a minimum if $\mathrm{F}^n(x_0)$ is positive.

In the application of this theory to questions of geometrical maxima and minima, it will subsequently appear that figs. 12 and 13 correspond to the analytical conditions of every derived-function vanishing, when $x = x_0$, up to one of an odd order inclusively, and of the next derived-function of an even order remaining finite; and figs. 14 and 15 correspond to the condition that the derived-function, which is the first not to vanish, is of an odd order.

Ex. 1.
$$\begin{aligned} \mathrm{F}(x) &= e^x + 2\cos x + e^{-x}, \\ \mathrm{F}'(x) &= e^x - 2\sin x - e^{-x}, = 0, \text{ if } x = 0, \\ \mathrm{F}''(x) &= e^x - 2\cos x + e^{-x}, = 0, \ldots \\ \mathrm{F}'''(x) &= e^x + 2\sin x - e^{-x}, = 0, \ldots \\ \mathrm{F}''''(x) &= e^x + 2\cos x + e^{-x}, = 4, \ldots \end{aligned}$$

$\therefore$ if $x = 0$, $\mathrm{F}(x) = 4$, which is a minimum value, as the fourth derived-function which is the first not to vanish is positive.

132.] We subjoin some examples of problems on maxima and minima, the study of which will be sufficient to enable the reader to apply the methods to other similar ones.

Ex. 1. To divide a given number a into two such parts that the product of the nth power of one, and the mth power of the other, may be a maximum.

Let a be the given number, of which let x be one part, therefore $a-x$ is the other; and we have

$$\mathrm{F}(x) = x^n (a-x)^m;$$

$$\begin{aligned} \therefore \quad \mathrm{F}'(x) &= x^{n-1} (a-x)^{m-1} (na-nx-mx), \\ &= (n+m)x^{n-1}(a-x)^{m-1}\left(\frac{na}{m+n} - x\right), \end{aligned}$$

$= 0$, if $x = 0$, and, if n be an even number, changes sign from $-$ to $+$, which indicates a minimum; but if n be odd, $\mathrm{F}'(x)$ does not change sign, and there is no corresponding maximum or minimum.

$= 0$, if $x = a$, and, if m be an even number, changes sign from $-$ to $+$, which indicates a minimum; but if m be odd, $F'(x)$ does not change sign, and there is no corresponding maximum or minimum.

$= 0$, if $x = \frac{na}{m+n}$, and changes sign from $+$ to $-$, which indicates a maximum value, viz. $F(x) = m^m n^n \left(\frac{a}{m+n}\right)^{m+n}$.

Ex. 2. To find the number which bears the least ratio to its logarithm.

Let x be the number;

$$\therefore \quad F(x) = \frac{x}{\log x}, \text{ which is to be a minimum,}$$

$$F'(x) = \frac{\log x - 1}{(\log x)^2},$$

$= 0$, when $x = e$, and changes sign from $-$ to $+$, which indicates a minimum, viz. $F(x) = e$.

Ex. 3. To divide a given number into x parts, so that their continued product may be a maximum.

Let a be the number, of which, as there are to be x equal parts, each part $= \frac{a}{x}$; and therefore, if $F(x) =$ the product of them,

$$F(x) = \left(\frac{a}{x}\right)^x,$$

$$\therefore \quad \operatorname{Log} F(x) = x\{\log a - \log x\},$$

$$\therefore \quad \frac{F'(x)}{F(x)} = \log a - \log x - 1,$$

$$= \log a - \log x - \log e,$$

$$= \log\left(\frac{a}{e}\right) - \log x;$$

$\therefore \quad F'(x) = 0$, if $x = \frac{a}{e}$, and changes sign from $+$ to $-$ which indicates a maximum; therefore each part $= e$, and the product of the whole $= (e)^{\frac{a}{e}}$.

Ex. 4. To inscribe in a given circle the greatest isosceles triangle.

Let the vertex of the triangle be at the extremity A of a diameter of the circle; let radius of circle $= a$; and let P and P′ be the other angular points of the triangle; therefore MP = MP′. Let

$$\left.\begin{array}{l} \text{CM} = x, \\ \text{MP} = y, \end{array}\right\} \quad \therefore \text{ equation to circle is } y^2+x^2 = a^2,$$

$$\begin{aligned} \text{Area of triangle} = \text{F}(x) &= \text{AM} \times \text{MP}, \\ &= (a+x)\,(a^2-x^2)^{\frac{1}{2}}, \end{aligned}$$

$$\therefore \quad \text{F}'(x) = \frac{a^2-ax-2x^2}{(a^2-x^2)^{\frac{1}{2}}} = \frac{(a+x)^{\frac{1}{2}}\,(a-2x)}{(a-x)^{\frac{1}{2}}},$$

$= 0$, if $x = \dfrac{a}{2}$, and the sign changes from $+$ to $-$, which indicates a maximum, of which the value is $\dfrac{3^{\frac{3}{2}}}{4}\,a^2$.

$\text{F}'(x)$ also has singular values when $x = +a$, and when $x = -a$, but as the sign of it in both cases passes from $\pm\sqrt{-}$ to $\pm$, and as the geometrical meaning of the values is plain enough, we must reserve the consideration of such critical values to a future part of the work.

Ex. 5. To inscribe the greatest rectangle in a semi-ellipse.

In fig. 17 let A′P′BPA be the given semi-ellipse, and MPP′M′ the rectangle inscribed in it, which is to be a maximum; let

$$\left.\begin{array}{l} \text{CM} = x, \\ \text{MP} = y, \end{array}\right\} \text{ and the equation to the ellipse be}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;$$

$$\begin{aligned} \therefore \quad \text{the rectangle} = \text{F}(x) &= \text{M}'\text{M} \times \text{MP}, \\ &= 2.\,\text{CM} \times \text{MP}, \\ &= 2\frac{b}{a}\,x\,(a^2-x^2)^{\frac{1}{2}}, \end{aligned}$$

$$\therefore \quad \text{F}'(x) = 2\frac{b}{a}\,\frac{a^2-2x^2}{(a^2-x^2)^{\frac{1}{2}}},$$

$= 0$, if $x = +\dfrac{a}{\sqrt{2}}$, and changes sign from $+$ to $-$, which indicates a maximum, and $\text{F}(x)$ becomes ab; and therefore the greatest rectangle is one-half of that contained by AA′ and CB.

Ex. 6. The whole surface of a cylinder being given, it is required to find its form when the content is a maximum.

Let $x =$ the radius of the base,

$y =$ the height of the cylinder;

$\therefore \quad \pi x^2 y =$ the content,

$2\pi x^2 + 2\pi xy =$ the whole surface.

Let the given surface be $2\pi a^2$, and the content be $F(x)$,

$$\therefore \quad 2\pi x^2 + 2\pi xy = 2\pi a^2$$

$$y = \frac{a^2 - x^2}{x},$$

$$F(x) = \pi x^2 y,$$

$$= \pi x(a^2 - x^2),$$

$$F'(x) = \pi(a^2 - 3x^2),$$

$= 0$, if $x = \frac{a}{\sqrt{3}}$, and changes sign from $+$ to $-$, $\therefore$ a maximum; in which case $y =$ the height $= 2\frac{a}{\sqrt{3}}$, and the content $= \frac{2\pi a^3}{3^{\frac{3}{2}}}$.

Ex. 7. Given the content of a cone; find its dimensions when the whole surface is a minimum.

Let $x =$ the radius of the base,

$y =$ the altitude of the cone;

therefore by Ex. 6, Art. 24,

$$\text{Content} = \frac{\pi x^2 y}{3},$$

$$\text{Whole surface} = \pi x^2 + \pi x(x^2 + y^2)^{\frac{1}{2}}.$$

Let the content $= \frac{\pi a^3}{3}$, and the surface be $F(x)$,

$$\therefore \quad \pi x^2 y = \pi a^3,$$

$$y = \frac{a^3}{x^2},$$

$$\text{F}(x) = \pi x^2 + \pi x (x^2 + y^2)^{\frac{1}{2}},$$

$$= \pi x^2 + \pi \frac{(a^6 + x^6)^{\frac{1}{2}}}{x},$$

$$\text{F}'(x) = 2\pi x - \pi \frac{a^6 - 2x^6}{x^2 (a^6 + x^6)^{\frac{1}{2}}},$$

$= 0$, if $x = \dfrac{a}{2^{\frac{1}{6}}}$, and changes sign from $-$ to $+$, which indicates a minimum; and $y = 2a$; in which case the semi-vertical angle $= \sin^{-1}\dfrac{1}{3}$, and the surface $= 2\pi a^2$.

Ex. 8. To describe the least cone about a given sphere.

In fig. 18 let the circle APBP′, and the triangle EFG, represent a plane section of the sphere and circumscribed cone, made by the paper passing through the sphere's centre; let radius of sphere $= a$, and CM $= x$, MP $= y$,

$$\therefore \quad x^2 + y^2 = a^2.$$

Then, by properties of the tangent of a circle,

$$\text{CE} = \frac{a^2}{x}, \quad \text{AF} = \frac{a(a+x)}{y} = a \left(\frac{a+x}{a-x}\right)^{\frac{1}{2}}.$$

Let content of cone $= \text{F}(x)$;

$$\therefore \quad \text{F}(x) = \frac{\pi}{3}\,\text{AF}^2 \times \text{AE} = \frac{\pi}{3}\, a^3 \frac{(a+x)^2}{ax - x^2};$$

$$\therefore \quad \text{F}'(x) = \frac{\pi a^4}{3} \frac{(a+x)(3x-a)}{(ax - x^2)^2},$$

$= 0$, if $x = \dfrac{a}{3}$, and changes sign from $-$ to $+$, $\therefore$ a minimum value, viz. $\dfrac{8}{3}\pi a^3$, and which is therefore twice the volume of the sphere. See Ex. 7, Art. 24.

Ex. 9. To cut the greatest parabola from a given cone.

In fig. 19,

$$\left.\begin{array}{ll}\text{Let radius of base of cone} & = b,\\ \text{Altitude of cone} & = a,\end{array}\right\} \quad \text{OM} = x, \ \text{MP} = y,$$

$$\text{Area of parabola} = \text{F}(x);$$

$$\therefore \quad \text{MQ}^2 = \text{OM} \times \text{MB}, \qquad \text{MP} = \text{MB}\,\frac{\text{CO}}{\text{OB}},$$

$$= x(2b-x), \qquad = \frac{(a^2+b^2)^{\frac{1}{2}}}{2b}\,(2b-x).$$

$$= 2bx-x^2,$$

By Ex. 8, Art. 24, area of parabola $= \dfrac{2}{3}\,\text{QQ}' \times \text{MP}$,

$$\therefore \quad \text{F}(x) = \frac{2}{3}\,2\,\text{MQ} \times \text{MP},$$

$$= \frac{4}{3}(2bx-x^2)^{\frac{1}{2}}\,\frac{(a^2+b^2)^{\frac{1}{2}}}{2b}\,(2b-x),$$

$$= \frac{2(a^2+b^2)^{\frac{1}{2}}}{3b}\,x^{\frac{1}{2}}\,(2b-x)^{\frac{3}{2}},$$

$$\therefore \quad \text{F}'(x) = \frac{2(a^2+b^2)^{\frac{1}{2}}}{3b}\,\frac{(2b-x)^{\frac{1}{2}}(b-2x)}{x^{\frac{1}{2}}},$$

$= 0$, and changes sign from $+$ to $-$, if $x = \dfrac{b}{2}$, $\therefore\ \text{OM} = \dfrac{\text{OA}}{2}$,

and the area of the greatest parabola $= \dfrac{3^{\frac{1}{2}}}{2}(a^2+b^2)^{\frac{1}{2}}\,b$.

SECTION 2.—*On Maxima and Minima of Implicit Functions of Two Variables.*

133.] Suppose it is required to find what values of x make y a maximum or minimum, when the equation connecting y and x is of the form

$$u = \text{F}(x, y) = c,$$

then, by Art. 48,

$$\frac{dy}{dx} = -\frac{\left(\dfrac{du}{dx}\right)}{\left(\dfrac{du}{dy}\right)},$$

and as the necessary condition of such a singular value of y is, that $\frac{dy}{dx}$ changes sign, and as a change of sign can only take place by $\frac{dy}{dx}$ being equal to 0 or ∞, it follows that either $\left(\frac{du}{dx}\right) = 0$ or ∞, or $\left(\frac{du}{dy}\right) = 0$ or ∞; but as $\frac{dy}{dx}$ must not be indeterminate in form, $\left(\frac{du}{dx}\right)$ and $\left(\frac{du}{dy}\right)$ must not simultaneously be equal either to 0 or to ∞.

Suppose that $\left(\frac{du}{dx}\right) = 0$, at a particular value for which $\left(\frac{du}{dy}\right)$ does not vanish or become infinite, then the critical value depends on the sign of $\frac{d^2y}{dx^2}$ (see Art. 131); being a maximum or minimum according as $\frac{d^2y}{dx^2}$ is positive or negative; and by Art. 78, equation (122), if $\left(\frac{du}{dx}\right) = 0$,

$$\frac{d^2y}{dx^2} = -\frac{\left(\frac{d^2u}{dx^2}\right)}{\left(\frac{du}{dy}\right)}.$$

Hence if this expression be positive there is a minimum, and if it be negative there is a maximum value.

This method however of determining maxima and minima is very incomplete, as it does not discuss the cases where $\left(\frac{du}{dy}\right)$, or any other of the partial derived-functions, becomes infinite; and it is therefore to be taken as a suggestion of the manner in which such problems are to be solved: the best plan is to determine the special maxima and minima values for each problem separately, as follows:

Ex. 1. Required to find the maxima and minima values of y, having given

$$y^3 + x^3 - 3axy = 0.$$

$$\frac{dy}{dx} = -\frac{x^2 - ay}{y^2 - ax},$$

which $= 0$, if $x^2 = ay$, i.e. if $y = \dfrac{x^2}{a}$, whereby we have from the equation

$$x^6 = 2a^3x^3;$$

$$\therefore \quad \left.\begin{array}{l} x = 0 \\ y = 0 \end{array}\right\}, \quad \text{or} \quad \left.\begin{array}{l} x = 2^{\frac{1}{3}} a \\ y = 2^{\frac{2}{3}} a \end{array}\right\}.$$

The latter values render the denominator of $\dfrac{dy}{dx}$ positive, and therefore there is a change of sign of $\dfrac{dy}{dx}$ from $+$ to $-$, and therefore these values correspond to a maximum.

If $x = 0$, $y = 0$, $\dfrac{dy}{dx} = \dfrac{0}{0}$, which must be evaluated as in Art. 114,

$$\frac{dy}{dx} = -\frac{2x - a\dfrac{dy}{dx}}{2y\dfrac{dy}{dx} - a} = -\frac{dy}{dx}, \text{ if } x = 0,\ y = 0,$$

$$\therefore \quad \frac{dy}{dx} = 0, \text{ or } = \infty.$$

If $\dfrac{dy}{dx} = 0$, it changes sign from $-$ to $+$, which indicates a minimum.

Section 3.—*Maxima and Minima of an Explicit Function of Two Independent Variables.*

134.] Let $u = F(x, y)$ be the explicit function of two independent variables x and y, of which the maxima and minima are to be determined.

Now observing the definition of maxima and minima, given in Art. 125, viz. that a maximum is greater, and a minimum less, than any and every value of the function in its immediate neighbourhood, it follows that if x_0 and y_0 are specific values of x and y, which give such a singular value to $F(x, y)$, then $F(x_0, y_0)$ is greater or less (as the case may be) than any value corresponding to the variables, whether x infinitesimally varies

while y does not vary, or whether y infinitesimally varies while x does not vary, or whether x and y are simultaneously increased, or whether as x increases y decreases, or *vice versâ*; which property may also be thus expressed: If $\text{F}(x_0, y_0)$ be a maximum or a minimum, $\text{F}(x_0 \pm h, y_0 \pm k)$ is less or greater than $\text{F}(x_0, y_0)$ whatever be the signs of h and k (which are infinitesimal increments) and in whatever manner the signs are combined; and also whether $k = 0$, when x is increased or diminished by h, and whether $h = 0$, when y is increased or diminished by k.

For $\text{F}(x, y)$ therefore to have such a singular value, it is necessary for x to be such as to make it a maximum or a minimum when y does not vary: which value of $\text{F}(x, y)$ we may call a *partial maximum or minimum* with respect to x.

Also it is necessary for y to be such as to make $\text{F}(x, y)$ a maximum or a minimum when x does not vary: and such may be called a *partial maximum or minimum* with respect to y.

And when these two *partial* maxima or minima are combined, then we shall have what may be aptly called a *total maximum or minimum*; but if a partial maximum or minimum is combined with a partial minimum or maximum, then we have not such a value as fulfils the requirements of our definition, for the singular value will not be greater or less than every other value in its immediate neighbourhood.

Now to express these conditions in mathematical language:

135.] It is plain from what has preceded in the first Section of the present Chapter, that the necessary condition of a partial maximum with respect to x are, that $\left(\frac{d\text{F}}{dx}\right) = 0$, and changes its sign from + to −, at the critical value of x.

Similarly for a partial maximum with respect to y, it is necessary that $\left(\frac{d\text{F}}{dy}\right) = 0$, and changes its sign from + to − at the critical value of y.

In a particular case the change of sign may be conveniently determined by an examination of the critical factors of $\left(\frac{d\text{F}}{dx}\right)$ and of $\left(\frac{d\text{F}}{dy}\right)$; and in the general case the following process supplies the required criteria:

First let us suppose that at the critical values of x and y $\left(\frac{d^2\text{F}}{dx^2}\right)$, $\left(\frac{d^2\text{F}}{dx\,dy}\right)$, $\left(\frac{d^2\text{F}}{dy^2}\right)$ do not all vanish, and let us consider the conditions for a maximum. Then, by Art. 130, if for particular values of the variables $\left(\frac{d\text{F}}{dx}\right)$ and $\left(\frac{d\text{F}}{dy}\right)$ vanish, and change sign from $+$ to $-$, $\frac{d}{dx}\left(\frac{d\text{F}}{dx}\right)$ and $\frac{d}{dy}\left(\frac{d\text{F}}{dy}\right)$ are both negative; but

$$\frac{d}{dx}\left(\frac{d\text{F}}{dx}\right) = \left(\frac{d^2\text{F}}{dx^2}\right) + \left(\frac{d^2\text{F}}{dx\,dy}\right)\frac{dy}{dx}; \tag{1}$$

$$\frac{d}{dy}\left(\frac{d\text{F}}{dy}\right) = \left(\frac{d^2\text{F}}{dx\,dy}\right)\frac{dx}{dy} + \left(\frac{d^2\text{F}}{dy^2}\right). \tag{2}$$

For convenience of writing, replace as follows:

$$\left(\frac{d^2\text{F}}{dx^2}\right) = \text{A}, \quad \left(\frac{d^2\text{F}}{dx\,dy}\right) = \text{B}, \quad \left(\frac{d^2\text{F}}{dy^2}\right) = \text{C}.$$

And to examine the signs of (1) and (2), let θ represent some quantity to which both may be equated; so that

$$\text{A}\,dx + \text{B}\,dy = \theta\,dx,$$

$$\text{B}\,dx + \text{C}\,dy = \theta\,dy;$$

which may be written

$$(\text{A} - \theta)\,dx + \text{B}\,dy = 0, \tag{3}$$

$$\text{B}\,dx + (\text{C} - \theta)\,dy = 0; \tag{4}$$

whence, eliminating dx and dy, we have the following quadratic in θ:

$$(\text{A} - \theta)(\text{C} - \theta) - \text{B}^2 = 0, \tag{5}$$

$$\theta^2 - (\text{A} + \text{C})\,\theta + \text{AC} - \text{B}^2 = 0; \tag{6}$$

of which the two roots are to be negative, since (1) and (2) must be negative, so that $\text{F}(x_0, y_0)$ may be a maximum; therefore all the coefficients must be positive, that is,

$$\text{A and C must both be negative,} \tag{7}$$

$$\text{and AC} - \text{B}^2 \text{ must be positive.} \tag{8}$$

Hence the necessary conditions that x_0, y_0 should render $\text{F}(x,y)$ a maximum are, that at the critical values

$$\left.\begin{array}{ll}(\alpha) & \left(\frac{d\text{F}}{dx}\right) = 0, \ \left(\frac{d\text{F}}{dy}\right) = 0, \\ (\beta) & \left(\frac{d^2\text{F}}{dx^2}\right) \text{ and } \left(\frac{d^2\text{F}}{dy^2}\right) \text{ must be negative,} \\ (\gamma) & \left(\frac{d^2\text{F}}{dx^2}\right)\left(\frac{d^2\text{F}}{dy^2}\right) - \left(\frac{d^2\text{F}}{dx\,dy}\right)^2 \text{must be positive.}\end{array}\right\} \quad (9)$$

This last condition having been first determined by Lagrange, is known by the name of Lagrange's condition.

136.] Similarly to determine a total minimum, it is requisite that it should be a partial minimum of a partial minimum, and therefore, supposing that $\left(\frac{d^2\text{F}}{dx^2}\right)$, $\left(\frac{d^2\text{F}}{dx\,dy}\right)$, $\left(\frac{d^2\text{F}}{dy^2}\right)$ do not all vanish at the critical values, that $\left(\frac{d\text{F}}{dx}\right) = 0$, and $\left(\frac{d\text{F}}{dy}\right) = 0$, and that $\frac{d}{dx}\left(\frac{d\text{F}}{dx}\right)$ and $\frac{d}{dy}\left(\frac{d\text{F}}{dy}\right)$ should be positive; accordingly, following the process of the last Article, the two values of θ, and therefore the roots of (6), must be positive; whence it follows, that A and C must be both positive, and that $\text{AC} - \text{B}^2$ must be positive; and therefore the conditions that x_0, y_0 should render $\text{F}(x, y)$ a minimum are, that

$$\left.\begin{array}{ll}(\alpha) & \left(\frac{d\text{F}}{dx}\right) = 0, \ \left(\frac{d\text{F}}{dy}\right) = 0, \\ (\beta) & \left(\frac{d^2\text{F}}{dx^2}\right) \text{ and } \left(\frac{d^2\text{F}}{dy^2}\right) \text{ must be positive,} \\ (\gamma) & \left(\frac{d^2\text{F}}{dx^2}\right)\left(\frac{d^2\text{F}}{dy^2}\right) - \left(\frac{d^2\text{F}}{dx\,dy}\right)^2 \text{must be positive.}\end{array}\right\} \quad (10)$$

137.] Ex. 1. To determine whether any, and what, values of x and y render $x^2y + xy^2 - axy$ a maximum or minimum.

$$\text{F}(x,y) = x^2y + xy^2 - axy,$$

$$\left(\frac{d\text{F}}{dx}\right) = 2xy + y^2 - ay = y(2x + y - a),$$

$$\left(\frac{d\text{F}}{dy}\right) = 2xy + x^2 - ax = x(2y + x - a),$$

$$\left(\frac{d^2\text{F}}{dx^2}\right) = 2y, \quad \left(\frac{d^2\text{F}}{dx\,dy}\right) = 2x + 2y - a, \quad \left(\frac{d^2\text{F}}{dy^2}\right) = 2x;$$

$\therefore$ putting $\left(\frac{d\text{F}}{dx}\right) = 0$, and $\left(\frac{d\text{F}}{dy}\right) = 0$, we have the following simultaneous values:

$$\left.\begin{array}{l} x = 0 \\ y = 0 \end{array}\right\} \quad \left.\begin{array}{l} x = a \\ y = 0 \end{array}\right\} \quad \left.\begin{array}{l} x = 0 \\ y = a \end{array}\right\} \quad \left.\begin{array}{l} x = \frac{a}{3} \\ y = \frac{a}{3} \end{array}\right\},$$

neither of the first three of which satisfy Lagrange's condition; and from the last we have $\left(\frac{d^2\text{F}}{dx^2}\right)$ and $\left(\frac{d^2\text{F}}{dy^2}\right)$ positive, and $\left(\frac{d^2\text{F}}{dx^2}\right)\left(\frac{d^2\text{F}}{dy^2}\right) - \left(\frac{d^2\text{F}}{dxdy}\right)^2 = \frac{a^2}{3}$;

$\therefore$ $x^2y + xy^2 - axy$ is a minimum, viz. $-\frac{a^3}{27}$, when $x = y = \frac{a}{3}$.

Ex. 2. To find a point within a triangle for which, if lines be drawn to the angles, the sum of their squares is a minimum.

Take A, one of the angular points (see fig. 20) of the triangle, for the origin, and let the base AB $= a$, and the coordinates to C be hk; and let the coordinates of P be x, y, and the sum of the squares $= \text{F}(x, y)$.

$$\begin{aligned} \therefore \quad \text{F}(x,y) &= x^2+y^2+(a-x)^2+y^2+(h-x)^2+(k-y)^2, \\ &= 3x^2+3y^2-2(a+h)\,x-2ky+a^2+h^2+k^2; \end{aligned}$$

$$\therefore \quad \left(\frac{d\text{F}}{dx}\right) = 6x-2(a+h) = 0, \text{ if } x = \frac{a+h}{3},$$

$$\left(\frac{d\text{F}}{dy}\right) = 6y-2k = 0, \text{ if } y = \frac{k}{3};$$

and as in both cases the change of sign is from $-$ to $+$, we have a partial minimum of a partial minimum, and therefore the necessary conditions of a total minimum.

Ex. 3. Inscribe the greatest triangle in a circle.

Fig. 21. Let one of the angular points of the triangle be at A, the extremity of the diameter ACB; and let P and Q be the other angles; let AC $=$ CB $= a$.

PAB $= \theta$, QAB $= \phi$; $\therefore$ by a property of the circle,

$$\left.\begin{aligned} \text{PA} &= 2a\cos\theta, \\ \text{QA} &= 2a\cos\phi; \end{aligned}\right\}$$

$\therefore$ Area of triangle $= F(\theta, \phi) = \frac{1}{2}$ PA $\times$ QA $\times \sin(\theta+\phi)$,

$$= 2a^2 \cos\theta \cos\phi \sin(\theta+\phi);$$

$$\therefore \left(\frac{dF}{d\theta}\right) = 2a^2 \cos\phi \{\cos\theta \cos(\theta+\phi) - \sin\theta \sin(\theta+\phi)\},$$

$$= 2a^2 \cos\phi \cos(2\theta+\phi),$$

$$\left(\frac{dF}{d\phi}\right) = 2a^2 \cos\theta \cos(2\phi+\theta);$$

$$\therefore \left(\frac{d^2F}{d\theta^2}\right) = -4a^2 \cos\phi \sin(2\theta+\phi),$$

$$\left(\frac{d^2F}{d\theta d\phi}\right) = -2a^2 \sin 2(\theta+\phi),$$

$$\left(\frac{d^2F}{d\phi^2}\right) = -4a^2 \cos\theta \sin(2\phi+\theta).$$

But if $\left(\frac{dF}{d\theta}\right) = 0$, and $\left(\frac{dF}{d\phi}\right) = 0$,

$$\left.\begin{aligned} 2\theta+\phi &= \frac{\pi}{2}, \\ 2\phi+\theta &= \frac{\pi}{2}, \end{aligned}\right\} \quad \therefore \quad \theta = \phi = \frac{\pi}{6}.$$

in which case $\left(\frac{d^2F}{d\theta^2}\right)$ and $\left(\frac{d^2F}{d\phi^2}\right)$ are both negative, and $\left(\frac{d^2F}{d\theta^2}\right)\left(\frac{d^2F}{d\phi^2}\right) - \left(\frac{d^2F}{d\theta d\phi}\right)^2 = 9a^4$; therefore the above critical values of θ and ϕ give a maximum: and as $\theta = \phi = \frac{\pi}{6}$, the triangle is equilateral. Hence the greatest triangle that can be inscribed in a circle is the equilateral one.

138.] In the case in which $\left(\frac{d^2F}{dx^2}\right)\left(\frac{d^2F}{dy^2}\right) - \left(\frac{d^2F}{dx dy}\right)^2$ is negative, the last term of the quadratic in θ of equation (6) is negative, and therefore the two values of θ are of different signs; whence, by means of (1) and (2), it follows that one of the partial singular values is a maximum, and the other is a

minimum: and therefore the conditions requisite for a total maximum or minimum are not fulfilled.

And if $\left(\frac{d^2\text{F}}{dx^2}\right)\left(\frac{d^2\text{F}}{dy^2}\right) = \left(\frac{d^2\text{F}}{dx\,dy}\right)^2$, then the last term of equation (6) $= 0$, and therefore one value of θ is zero; and therefore either (1) or (2) $= 0$, and therefore either $\left(\frac{d\text{F}}{dx}\right)$ or $\left(\frac{d\text{F}}{dy}\right)$ undergoes no variation; whereas then there is a partial maximum or minimum with respect to one of the variables, the other is such that corresponding to its variations the function is constant; hence we have a *locus* of such partial maxima or minima.

These several conditions will be more clearly understood from the geometrical analogues of them in the theory of curved surfaces; which however it would be premature to explain in this place, and therefore we reserve them until they naturally arise in the course of the Treatise.

139.] If the critical values x_0, y_0 which are deduced from $\left(\frac{d\text{F}}{dx}\right) = 0$, and $\left(\frac{d\text{F}}{dy}\right) = 0$, also make $\left(\frac{d^2\text{F}}{dx^2}\right) = 0$, $\left(\frac{d^2\text{F}}{dx\,dy}\right) = 0$, $\left(\frac{d^2\text{F}}{dy^2}\right) = 0$, then, in the general case, we must, by means of equation (37), Art. 122, have recourse to an artifice analogous to that of Art. 131; and therefore, as the sign of $\text{F}(x_0+h, y_0+k) - \text{F}(x_0, y_0)$ must not change, whatever be the signs of h and k, it follows that the term in the expansion of equation (37), Art. 122, which is the first not to vanish at the critical values, must involve h and k to even powers. Therefore if $\left(\frac{d^2\text{F}}{dx^2}\right) = \left(\frac{d^2\text{F}}{dx\,dy}\right) = \left(\frac{d^2\text{F}}{dy^2}\right) = 0$, it is necessary also that $\left(\frac{d^3\text{F}}{dx^3}\right) = \ldots = \left(\frac{d^3\text{F}}{dy^3}\right) = 0$, and that $\left(\frac{d^4\text{F}}{dx^4}\right), \ldots . \left(\frac{d^4\text{F}}{dy^4}\right)$ should not all vanish; also amongst these last terms a relation must exist analogous to Lagrange's condition. In a particular case however the change of sign of $\left(\frac{d\text{F}}{dx}\right)$ and of $\left(\frac{d\text{F}}{dy}\right)$ at the critical values may generally be detected.

Section 4.—*Maxima and Minima of Functions of Three and more Independent Variables.*

140.] Firstly, consider a function of three independent variables, x, y, z, of the form

$$u = \text{F}(x, y, z).$$

Extending the principles of Art. 134 to this more general case, it appears that a total maximum or minimum of a function of three variables must arise from the combination of three several partial maxima or minima with respect to the several variables. And also, as any two of the three variables may vary, while the remaining one does not vary, that the conditions of such a combination of two partial maxima or minima must be fulfilled. Which conditions are,

$$\left(\frac{d\text{F}}{dx}\right) = 0, \quad \left(\frac{d\text{F}}{dy}\right) = 0, \quad \left(\frac{d\text{F}}{dz}\right) = 0; \tag{11}$$

$$\left.\begin{array}{l} \left(\frac{d^2\text{F}}{dy^2}\right)\left(\frac{d^2\text{F}}{dz^2}\right) - \left(\frac{d^2\text{F}}{dy\,dz}\right)^2 > 0; \quad \left(\frac{d^2\text{F}}{dz^2}\right)\left(\frac{d^2\text{F}}{dx^2}\right) - \left(\frac{d^2\text{F}}{dz\,dx}\right)^2 > 0; \\ \left(\frac{d^2\text{F}}{dx^2}\right)\left(\frac{d^2\text{F}}{dy^2}\right) - \left(\frac{d^2\text{F}}{dx\,dy}\right)^2 > 0; \end{array}\right\} \tag{12}$$

therefore $\left(\frac{d^2\text{F}}{dx^2}\right)$, $\left(\frac{d^2\text{F}}{dy^2}\right)$, $\left(\frac{d^2\text{F}}{dz^2}\right)$ must be of the same sign; and the singular value of $\text{F}(x, y, z)$ is a maximum or minimum, according as they are negative or positive. There is also another relation between the several partial second-derived-functions; for since $\frac{d}{dx}\left(\frac{d\text{F}}{dx}\right)$, $\frac{d}{dy}\left(\frac{d\text{F}}{dy}\right)$, and $\frac{d}{dz}\left(\frac{d\text{F}}{dz}\right)$ must be of the same sign, and be negative for a maximum value, and positive for a minimum; and since

$$\frac{d}{dx}\left(\frac{d\text{F}}{dx}\right) = \left(\frac{d^2\text{F}}{dx^2}\right) + \left(\frac{d^2\text{F}}{dx\,dy}\right)\frac{dy}{dx} + \left(\frac{d^2\text{F}}{dx\,dy}\right)\frac{dz}{dx}; \tag{13}$$

$$\frac{d}{dy}\left(\frac{d\text{F}}{dy}\right) = \left(\frac{d^2\text{F}}{dy\,dx}\right)\frac{dx}{dy} + \left(\frac{d^2\text{F}}{dy^2}\right) + \left(\frac{d^2\text{F}}{dy\,dz}\right)\frac{dz}{dy}; \tag{14}$$

$$\frac{d}{dz}\left(\frac{d\text{F}}{dz}\right) = \left(\frac{d^2\text{F}}{dx\,dz}\right)\frac{dx}{dz} + \left(\frac{d^2\text{F}}{dy\,dz}\right)\frac{dy}{dz} + \left(\frac{d^2\text{F}}{dz^2}\right); \tag{15}$$

Now for convenience of writing, let

$$\left(\frac{d^2 F}{dx^2}\right) = A, \quad \left(\frac{d^2 F}{dy^2}\right) = B, \quad \left(\frac{d^2 F}{dz^2}\right) = C,$$

$$\left(\frac{d^2 F}{dy\,dz}\right) = E, \quad \left(\frac{d^2 F}{dz\,dx}\right) = F, \quad \left(\frac{d^2 F}{dx\,dy}\right) = G,$$

and as (13), (14), (15) are to be of the same sign, let θ be the symbol for *some* quantity which has the same sign in all; then the following system results:

$$\left.\begin{array}{r} (A-\theta)\,dx + G\,dy + F\,dz = 0 \\ G\,dx + (B-\theta)\,dy + E\,dz = 0 \\ F\,dx + E\,dy + (C-\theta)\,dz = 0 \end{array}\right\}; \qquad (16)$$

whence, by cross-multiplication,

$$(A-\theta)\,(B-\theta)\,(C-\theta) - E^2(A-\theta) - F^2(B-\theta) - G^2(C-\theta) + 2\,EFG = 0, \qquad (17)$$

the common Discriminating Cubic (as it is called), and which has three real roots; and, when expanded, becomes

$$\theta^3 - (A+B+C)\theta^2 + (BC+CA+AB-E^2-F^2-G^2)\theta - (ABC+2EFG-AE^2-BF^2-CG^2) = 0. \qquad (18)$$

Of which equation the three roots are to be of the same sign, and the result is a maximum or a minimum, according as they are negative or positive; therefore, besides the former conditions, we must have the following expression negative for a maximum and positive for a minimum, viz.:

$$ABC+2EFG-AE^2-BF^2-CG^2. \qquad (19)$$

Hence, that a function of three variables may have a maximum or a minimum value, the critical values must satisfy $3 + 3 + 1 \,(= 7)$ conditions, viz. three of equations (11), three of equations (12), and one of equation (19).

141.] Lastly, let us consider the general case, and let

$$F(x_1, x_2 \ldots\ldots x_n)$$

be a function of n independent variables, of which the maxima and minima are to be determined; and, for convenience of notation, let

$$\left(\frac{d^2F}{dx_1^2}\right) = (1,1), \left(\frac{d^2F}{dx_1\,dx_2}\right) = (1,2), \ldots \left(\frac{d^2F}{dx_1\,dx_n}\right) = (1,n),$$

$$\left(\frac{d^2F}{dx_2\,dx_1}\right) = (2,1), \quad \left(\frac{d^2F}{dx_2^2}\right) = (2,2), \ldots \left(\frac{d^2F}{dx_2\,dx_n}\right) = (2,n),$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\left(\frac{d^2F}{dx_n\,dx_1}\right) = (n,1), \left(\frac{d^2F}{dx_n\,dx_2}\right) = (n,2), \ldots \quad \left(\frac{d^2F}{dx_n^2}\right) = (n,n).$$

Of which it is to be observed, that

$$(1,2) = (2,1), \ldots (n,1) = (1,n), \ldots$$

then we have the following equations:

$$\left.\begin{array}{l} \{(1,1)-\theta\}\,dx_1 + (1,2)\,dx_2 + \ldots\ldots + (1,n)\,dx_n = 0 \\ (2,1)\,dx_1 + \{(2,2)-\theta\}\,dx_2 + \ldots\ldots + (2,n)\,dx_n = 0 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ (n,1)\,dx_1 + (n,2)\,dx_2 + \ldots\ldots + \{(n,n)-\theta\}\,dx_n = 0 \end{array}\right\} \quad (20)$$

Whence, by the elimination of the n quantities, $dx_1, dx_2 \ldots dx_n$, there will result an equation in θ of n dimensions, all the roots of which are to be of the same sign; and according as they are positive or negative, will the corresponding value of the function be a minimum or maximum.

The number of conditions which are required to be fulfilled may thus be found: As the total maximum or minimum arises from the combination of the several partial singular values, all the conditions which they involve must separately be satisfied. Hence it is easy to see, that when the equation in θ has been formed, there will be involved, and to be satisfied, in the coefficients of

θ^{n-1}, n conditions;

θ^{n-2}, $\dfrac{n(n-1)}{1.2}$ . . . . . .

θ^{n-3}, $\dfrac{n(n-1)(n-2)}{1.2.3}$ . . . . . .

. . . .

θ, ... n . . . . . .

in the constant term, . . . 1 . . . . . .

$$\text{the sum of which} = n + \frac{n(n-1)}{1.2} + \frac{n(n-1)(n-2)}{1.2.3} + \ldots + n + 1,$$
$$= (1+1)^n - 1,$$
$$= 2^n - 1^*.$$

SECTION 5.—*On Maxima and Minima of Functions of Variables when all are not Independent.*

142.] A problem, which frequently occurs, is the determination of maxima and minima of functions of many variables, when certain relations between the variables are given, so that all those involved in the original functional equation are not independent. Thus suppose that we have to determine the maxima and minima values of

$$u = \mathrm{F}(x, y, z \ldots .), \tag{21}$$

which is a function of n variables; and suppose besides that there are given m equations connecting these variables, viz.:

$$\left.\begin{array}{l} \mathrm{F}_1(x, y, z \ldots\ldots) = 0, \\ \mathrm{F}_2(x, y, z \ldots\ldots) = 0, \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \mathrm{F}_m(x, y, z \ldots\ldots) = 0. \end{array}\right\} \tag{22}$$

In order to apply the method which has been explained in the last Article, it would be necessary to eliminate m variables between $m+1$ equations, by which means u would become a function of $n-m$ variables, all of which would be independent of each other; and then forming the partial derived-functions $\left(\frac{d\mathrm{F}}{dx}\right)$, $\left(\frac{d\mathrm{F}}{dy}\right)$, $\left(\frac{d\mathrm{F}}{dz}\right)$ ......, the number of which is $n-m$, and equating each to 0, there would be $n-m$ equations, from which we could (theoretically at least) determine the $n-m$ variables. This method however, though theoretically possible, is fre-

* On the necessity and sufficiency of conditions equivalent to those determined in the text above, in order that an equation of n dimensions may have all positive roots, see "Elementary Theorems relating to Determinants," by William Spottiswoode, M.A., of Balliol College, Oxford, (Longmans, London, 1851,) pp. 35—37, in which however there is an error, as the general term of the series of required conditions, instead of the sum of the series as above, is given as the whole number of conditions.

quently attended with great difficulty on account of elimination; and, if the original expressions be symmetrical, the symmetry is destroyed by it; in which case it is better to proceed as follows:

It is plain from what has been said, that as there are $n-m$ variables entirely independent in their variations, we have $n-m$ conditions to make; which will be equivalent to equating to 0 the $(n-m)$ partial derived-functions, with respect to these variables of $\mathrm{F}(x, y, z \ldots\ldots)$. Differentiating the functions in order, and remembering that $\mathrm{D}u = 0$, because u is to be a maximum or a minimum, we have

$$\left.\begin{aligned}
\mathrm{D}u = 0 &= \left(\frac{d\mathrm{F}}{dx}\right)dx + \left(\frac{d\mathrm{F}}{dy}\right)dy + \left(\frac{d\mathrm{F}}{dz}\right)dz + \ldots\ldots \\
0 &= \left(\frac{d\mathrm{F}_1}{dx}\right)dx + \left(\frac{d\mathrm{F}_1}{dy}\right)dy + \left(\frac{d\mathrm{F}_1}{dz}\right)dz + \ldots\ldots \\
0 &= \left(\frac{d\mathrm{F}_2}{dx}\right)dx + \left(\frac{d\mathrm{F}_2}{dy}\right)dy + \left(\frac{d\mathrm{F}_2}{dz}\right)dz + \ldots\ldots \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
0 &= \left(\frac{d\mathrm{F}_m}{dx}\right)dx + \left(\frac{d\mathrm{F}_m}{dy}\right)dy + \left(\frac{d\mathrm{F}_m}{dz}\right)dz + \ldots\ldots
\end{aligned}\right\} \quad (23)$$

The meaning of which is, that $x, y, z \ldots\ldots$, do not vary independently of each other, but consistently with the conditions involved in the last m equations. Hence to eliminate $dx, dy, dz \ldots\ldots$, multiply these equations by indeterminate quantities, $\lambda_1, \lambda_2, \lambda_3 \ldots\ldots \lambda_m$, and add to the first; and collecting the coefficients of $dx, dy, dz \ldots\ldots$, we have

$$\begin{aligned}
&\left\{\left(\frac{d\mathrm{F}}{dx}\right) + \lambda_1\left(\frac{d\mathrm{F}_1}{dx}\right) + \lambda_2\left(\frac{d\mathrm{F}_2}{dx}\right) + \ldots\ldots + \lambda_m\left(\frac{d\mathrm{F}_m}{dx}\right)\right\} dx \\
+ &\left\{\left(\frac{d\mathrm{F}}{dy}\right) + \lambda_1\left(\frac{d\mathrm{F}_1}{dy}\right) + \lambda_2\left(\frac{d\mathrm{F}_2}{dy}\right) + \ldots\ldots + \lambda_m\left(\frac{d\mathrm{F}_m}{dy}\right)\right\} dy \\
+ &\left\{\left(\frac{d\mathrm{F}}{dz}\right) + \lambda_1\left(\frac{d\mathrm{F}_1}{dz}\right) + \lambda_2\left(\frac{d\mathrm{F}_2}{dz}\right) + \ldots\ldots + \lambda_m\left(\frac{d\mathrm{F}_m}{dz}\right)\right\} dz \\
+ &\left\{ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \right\} \ldots \\
+ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad = 0,
\end{aligned}$$

Which equation is subject to n conditions, viz. $n-m$, on account of $n-m$ independent variables being involved, and m on account of our having introduced m indeterminate multipliers $\lambda_1, \lambda_2, \lambda_3 \ldots\ldots \lambda_m$. Let these conditions be, that the coefficient of each differential is equal to 0; therefore

$$\left.\begin{array}{l} \left(\frac{dF}{dx}\right) + \lambda_1\left(\frac{dF_1}{dx}\right) + \lambda_2\left(\frac{dF_2}{dx}\right) + \ldots\ldots + \lambda_m\left(\frac{dF_m}{dx}\right) = 0 \\ \left(\frac{dF}{dy}\right) + \lambda_1\left(\frac{dF_1}{dy}\right) + \lambda_2\left(\frac{dF_2}{dy}\right) + \ldots\ldots + \lambda_m\left(\frac{dF_m}{dy}\right) = 0 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}\right\} \quad (24)$$

between which equations $\lambda_1, \lambda_2 \ldots\ldots \lambda_m$ are to be eliminated, and $x, y, z \ldots\ldots$ determined, which will be the values corresponding to a maximum or a minimum value of $F(x, y, z \ldots\ldots)$.

The sign of the second differential coefficient will determine whether the particular value be a maximum or a minimum: but in most cases where this method is applicable, the form of the function at once decides whether it admits of a maximum or of a minimum.

143.] In the case in which only one equation is given connecting the n variables involved in the given function, whose maximum or minimum value is to be determined, the above results assume a particular form, by means of which the process is much simplified.

Let
$$u = F(x, y, z \ldots\ldots) \qquad (25)$$

be the function of which the maximum or minimum value is to be determined; and suppose the variables to be subject to the relation expressed by the equation

$$f(x, y, z \ldots\ldots) = c; \qquad (26)$$

then, differentiating (25) and (26), and putting $Du = 0$, by reason of its being a maximum or a minimum, we have

$$DF = 0 = \left(\frac{dF}{dx}\right)dx + \left(\frac{dF}{dy}\right)dy + \left(\frac{dF}{dz}\right)dz + \ldots\ldots \qquad (27)$$

$$Df = 0 = \left(\frac{df}{dx}\right)dx + \left(\frac{df}{dy}\right)dz + \left(\frac{df}{dz}\right)dz + \ldots\ldots \qquad (28)$$

whence, multiplying (27) by an indeterminate multiplier λ, and subtracting (28) from it, we have

$$\left\{\lambda\left(\frac{d\mathrm{F}}{dx}\right)-\left(\frac{df}{dx}\right)\right\}dx+\left\{\lambda\left(\frac{d\mathrm{F}}{dy}\right)-\left(\frac{df}{dy}\right)\right\}dy+\ldots=0; \quad (29)$$

and by virtue of the argument of the last Article, equating to zero the several coefficients of dx, dy, dz ......, we have

$$\frac{1}{\lambda}=\frac{\left(\frac{d\mathrm{F}}{dx}\right)}{\left(\frac{df}{dx}\right)}=\frac{\left(\frac{d\mathrm{F}}{dy}\right)}{\left(\frac{df}{dy}\right)}=\frac{\left(\frac{d\mathrm{F}}{dz}\right)}{\left(\frac{df}{dz}\right)}=\ldots\ldots \quad (30)$$

that is, the ratio of the coefficients of the same differentials in (27) and (28) is constant.

The algebraical criterion for discriminating between a maximum and a minimum is too complicated to be of any service even in this particular case.

144.] Ex. 1. To determine the minimum value of $x^2+y^2+z^2$, having given $p = ax+by+cz$, where p, a, b, c are constants.

Let
$$u^2 = x^2+y^2+z^2,$$
$$p = ax+by+cz;$$
$$\therefore \quad u\mathrm{D}u = xdx+ydy+zdz = 0,$$
$$0 = adx+bdy+cdz;$$

$\therefore$ by reason of the last Article,

$$\frac{x}{a}=\frac{y}{b}=\frac{z}{c}=\frac{x^2+y^2+z^2}{ax+by+cz}=\frac{u^2}{p}$$

$$=\frac{\{x^2+y^2+z^2\}^{\frac{1}{2}}}{\{a^2+b^2+c^2\}^{\frac{1}{2}}}=\frac{u}{\{a^2+b^2+c^2\}^{\frac{1}{2}}}, \text{ by Preliminary Theorem I;}$$

$$\therefore \quad u=\frac{p}{\{a^2+b^2+c^2\}^{\frac{1}{2}}},$$

$$x=\frac{ap}{a^2+b^2+c^2}, \quad y=\frac{bp}{a^2+b^2+c^2}, \quad z=\frac{cp}{a^2+b^2+c^2}.$$

The above is the solution of the problem, To find the shortest distance from the origin to a given plane.

Ex. 2. To determine the greatest triangle of given perimeter.

Let the area of the triangle $= u$, and the sides be x, y, z, and the given perimeter $= 2S$.

$$\therefore \quad 2S = x + y + x,$$

$$u^2 = S(S-x)(S-y)(S-z);$$

$\therefore$ differentiating the former equation, and taking the logarithmic differential of the second,

$$0 = dx + dy + dz,$$

$$-\frac{2\,du}{u} = \frac{dx}{S-x} + \frac{dy}{S-y} + \frac{dz}{S-z} = 0;$$

$$\therefore \quad S - x = S - y = S - z,$$

$$\therefore \quad x = y = z = \frac{2S}{3},$$

$$\text{Area of triangle} = \frac{S^2}{3^{\frac{3}{2}}};$$

that is, the triangle must be equilateral.

In precisely the same manner may it be shewn that of all figures of a given number of sides and of given perimeter, the equilateral and equiangular one is the greatest.

Hence also it follows that, of all plane figures of given perimeter, the circle is that whose area is the greatest.

For let $2S$ be the given perimeter, and n be the required number of sides,

$$\therefore \quad \text{each side} = \frac{2S}{n},$$

$$\text{and} \qquad \therefore \quad \text{area} = \frac{S^2}{n} \cot \frac{\pi}{n},$$

$$= \frac{S^2}{\pi}\left(\frac{\pi}{n} \cot \frac{\pi}{n}\right).$$

But the last factor continually increases as $\frac{\pi}{n}$ decreases, and attains a maximum when $n = \infty$; in which case the polygon becomes a circle, which is therefore the greatest of all plane figures under a given perimeter.

Ex. 3. To divide a number a into three parts x, y, z, so that $x^m\, y^n\, z^p$ may be a maximum.

$$u = x^m\, y^n\, z^p, \quad \therefore \ \log u = m \log x + n \log y + p \log z;$$

$$a = x + y + z;$$

$$\therefore \ \frac{\mathrm{D}u}{u} = 0 = \frac{m\,dx}{x} + \frac{n\,dy}{y} + \frac{p\,dz}{z},$$

$$0 = dx + dy + dz;$$

$$\therefore \ \frac{m}{x} = \frac{n}{y} = \frac{p}{z} = \frac{m+n+p}{x+y+z} = \frac{m+n+p}{a},$$

$$\therefore \ x = \frac{am}{m+n+p}, \quad y = \frac{an}{m+n+p}, \quad z = \frac{ap}{m+n+p},$$

$$u = m^m\, n^n\, p^p \left(\frac{a}{m+n+p}\right)^{m+n+p}.$$

Ex. 4. To determine the greatest quadrilateral figure which can be contained by four given straight lines a, b, c, e.

In fig. 22. Let AB $= a$, BC $= b$, CD $= c$, DA $= e$; the angle ABC $= \theta$, ADC $= \phi$; $u =$ area.

$$\therefore \ u = \frac{1}{2}\left\{ab \sin\theta + ce \sin\phi\right\},$$

$$(\mathrm{AC})^2 = a^2 + b^2 - 2ab\cos\theta = c^2 + e^2 - 2ce\cos\phi;$$

$$\therefore \ \mathrm{D}u = 0 = \frac{1}{2}\left\{ab\cos\theta\, d\theta + ce\cos\phi\, d\phi\right\},$$

$$0 = 2ab\sin\theta\, d\theta - 2ce\sin\phi\, d\phi;$$

$$\therefore \ \frac{\cos\theta}{\sin\theta} = -\frac{\cos\phi}{\sin\phi},$$

$$\therefore \ \tan\theta = -\tan\phi,$$

$$\therefore \ \theta = 180^\circ - \phi;$$

that is, $\theta + \phi = 180^\circ$, and the opposite angles of the quadrilateral figure are together equal to two right angles, and the quadrilateral is such as can be inscribed in a circle.

Ex. 5. To determine the maximum and minimum values of the central radii vectores of an ellipse, and their relation to a pair of conjugate axes.

Let the ellipse be referred to a pair of conjugate diameters, whose lengths are $2a_1$ and $2b_1$, as the coordinate axes; and let ω be the angle between the axes, and r be the length of any central radius vector; then the equation to the ellipse is

$$\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} = 1.$$

Also
$$r^2 = x^2 + 2xy\cos\omega + y^2;$$

$$\therefore \quad \frac{x}{a_1^2}\,dx + \frac{y}{b_1^2}\,dy = 0,$$

$$(x+y\cos\omega)\,dx + (y+x\cos\omega)\,dy = r\,dr = 0;$$

therefore employing an indeterminate multiplier, we have

$$\left\{\frac{x}{a_1^2} + \lambda(x+y\cos\omega)\right\}dx + \left\{\frac{y}{b_1^2} + \lambda(y+x\cos\omega)\right\}dy = 0;$$

And equating to 0 the coefficients of dx and dy,

$$\frac{x}{a_1^2} + \lambda(x+y\cos\omega) = 0,$$

$$\frac{y}{b_1^2} + \lambda(y+x\cos\omega) = 0;$$

whence, multiplying the former by x and the latter by y, and adding

$$\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} + \lambda(x^2 + 2xy\cos\omega + y^2) = 0,$$

$$\therefore \quad 1 + \lambda r^2 = 0, \qquad \therefore \quad \lambda = -\frac{1}{r^2};$$

and substituting

$$\left(\frac{1}{a_1^2} - \frac{1}{r^2}\right)x - \frac{\cos\omega}{r^2}\,y = 0,$$

$$-\frac{\cos\omega}{r^2}\,x + \left(\frac{1}{b_1^2} - \frac{1}{r^2}\right)y = 0;$$

whence, by cross-multiplication,

$$\left(\frac{1}{a_1^2} - \frac{1}{r^2}\right)\left(\frac{1}{b_1^2} - \frac{1}{r^2}\right) - \frac{1}{r^4}(\cos\omega)^2 = 0;$$

$$\therefore \quad r^4 - (a_1^2 + b_1^2)\,r^2 + a_1^2 b_1^2 (\sin\omega)^2 = 0.$$

And let a and b be the greatest and least values of the ra vectores, so that a^2 and b^2 will be the roots of the last equatio then by the theory of equations

$$a^2 + b^2 = a_1^2 + b_1^2,$$
$$a^2 b^2 = a_1^2 b_1^2 (\sin \omega)^2,$$

by means of which the values of a and b may be easily termined.

By a similar process may we determine the analogous re tions of the principal axes of an ellipsoid to any system conjugate axes.

Ex. 6. To inscribe in a sphere the greatest parall epipedon.

Let $x^2 + y^2 + z^2 = a^2$ be the equation to the sphere; a let u be the content of the parallelepipedon.

$$\therefore \quad u = 8xyz;$$

$\therefore$ taking the logarithmic differential,

$$\frac{\mathrm{D}u}{u} = 0 = \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}.$$

Also
$$0 = x\,dx + y\,dy + z\,dz.$$

$$\therefore \quad x^2 = y^2 = z^2 = \frac{a^2}{3},$$

$$\therefore \quad x = y = z = \frac{a}{\sqrt{3}},$$

$$\text{and volume of parallelepipedon} = \frac{8a^3}{3^{\frac{3}{2}}}.$$

Ex. 7. To find a point within a triangle, such that t sum of the lines drawn from it to the angular points may a minimum.

In fig. 20. Let P be the required point; and let AP = BP $= y$, CP $= z$, BC $= a$, CA $= b$, AB $= c$, BPC $= \theta$, CPA = APB $= \psi$; $u =$ sum of the required lines; wherefore we have

$$u = x + y + z; \tag{3}$$

$$\left.\begin{aligned} a^2 &= y^2 - 2yz\cos\theta + z^2, \\ b^2 &= z^2 - 2zx\cos\phi + x^2, \\ c^2 &= x^2 - 2xy\cos\psi + y^2; \end{aligned}\right\} \tag{3}$$

$$\theta + \phi + \psi = 2\pi. \tag{3}$$

Whence, by differentiation,

$$\mathrm{D}u = 0 = dx + dy + dz; \tag{34}$$

$$\left.\begin{aligned} 0 &= (y - z\cos\theta)\,dy + (z - y\cos\theta)\,dz + yz\sin\theta\,d\theta, \\ 0 &= (z - x\cos\phi)\,dz + (x - z\cos\phi)\,dx + zx\sin\phi\,d\phi, \\ 0 &= (x - y\cos\psi)\,dx + (y - x\cos\psi)\,dy + xy\sin\psi\,d\psi; \end{aligned}\right\} \tag{35}$$

$$0 = d\theta + d\phi + d\psi. \tag{36}$$

Now observing that we have six variables involved, which are connected by four equations (32) and (33), there are two independent ones; multiplying therefore the equations (35) severally by x, y, z, and adding, we have

$$\begin{aligned} \{x(y+z) - yz(\cos\phi + \cos\psi)\}\,dx + \{y(z+x) - zx(\cos\psi + \cos\theta)\}\,dy \\ + \{z(x+y) - xy(\cos\theta + \cos\phi)\}\,dz \\ + xyz\{\sin\theta\,d\theta + \sin\phi\,d\phi + \sin\psi\,d\psi\} = 0. \end{aligned} \tag{37}$$

Whence, in a manner similar to that of Art. 143, multiplying (34) and (36) by two indeterminate multipliers, and adding them to (37), and equating to zero the coefficients of the variables, we have

$$\sin\theta = \sin\phi = \sin\psi;$$

$$\therefore\ \theta = \phi = \psi = 120^{\circ}.$$

Let therefore three segments of circles be described on the sides of the triangle, each containing an angle of 120°; these will meet in a point, which point will be that required.

CHAPTER VIII.

APPLICATION OF THE PRECEDING PRINCIPLES TO THE DETERMINATION OF SOME QUESTIONS OF PURE ALGEBRA.

145.] In the present Chapter we take the following to be the type of algebraical equations of the nth degree, and for convenience of reference symbolize it by $f(x)$,

$$x^n - p_1 x^{n-1} + p_2 x^{n-2} - p_3 x^{n-3} + \ldots \pm p_{n-1} x \mp p_n = 0 = f(x), \quad (1)$$

the upper or lower sign being taken according as n is odd or even, n being integral and positive, and no fractional or negative powers of x being involved in the equation; $p_1\, p_2 \ldots\ldots p_n$ being constant coefficients, real or imaginary, and the coefficient of the highest power of x being unity.

A *root* of such an equation is a value which, when substituted for the unknown quantity x, makes the whole to vanish; thus, if a be a root of $f(x)$,

$$f(a) = 0.$$

An imaginary or impossible expression is one of the form

$$a + b\sqrt{-1}, \quad (2)$$

and which, as was explained in Art. 57, always admits of being put in the form
$$r(\cos\theta + \sqrt{-1}\sin\theta),$$

of which r is called the Modulus, and

$$r^2 = a^2 + b^2 = (a + \sqrt{-1}\,b)\,(a - \sqrt{-1}\,b), \quad (3)$$

the latter pair of imaginary factors being called conjugate to each other.

Now of such general algebraical expressions as (1), almost all the properties which are proved in the usual text-books on the Theory of Equations arise from considering them in their *resolved* or *analytical* form, that is, as made up of factors of the form $x - a,\ x - b, \ldots\ldots x - a + \beta\sqrt{-1},\ x - a - \beta\sqrt{-1} \ldots\ldots;$

e the relations which exist between the roots and the
nts, between two equations the roots of one of which
nmetrical functions of those of the other, &c. But a
treatment of the subject requires the equations to be
red in their *synthetical* or unresolved state, and we pro-
the following Articles to exhibit such properties of them
compounded state, as fall within the grasp of the pre-
principles; and the first question that meets us is, Has
equation a root? or in other words, Is there a value,
imaginary, which, when substituted for x in $f(x)$, will
$f(x) = 0$?
e thought that an unfair assumption is made in the follow-
icles in the extension to impossible quantities of processes
rentiation which have been hitherto applied to possible
ies, let it be borne in mind that the principles of differ-
n as unfolded in the first and second Chapters require
e law of continuity to be satisfied in the functions to
they are applied, and that impossible quantities satisfy
w as well as possible ones. And I would also observe
logarithmic and circular functions are related to each
by means of the exponential equivalents, which involve
s of impossibility, and as we have introduced and ope-
n each of these in the previous Chapters, our processes
een applied to impossible as well as to possible quantities.

] THEOREM I.—If $f(x)$ be a function of x of the form
ere are always one or more values of x of the form
$\sqrt{-1}$, which, when substituted for x, will render $f(x) = 0$;
ther words, Every equation has a root.

tly, we must shew that when for x we substitute in $f(x)$
ost general form $y + z\sqrt{-1}$, $f(y + z\sqrt{-1})$ does not
of an absolute maximum or minimum value.

anding by Taylor's Series $f(y + z\sqrt{-1})$, we have

$$z\sqrt{-1}) = f(y) + \sqrt{-1}\,f'(y)\,\frac{z}{1} - f''(y)\,\frac{z^2}{1.2}$$

$$- \sqrt{-1}\,f'''(y)\,\frac{z^3}{1.2.3} + f''''(y)\,\frac{z^4}{1.2.3.4} + \dots\dots \quad (4)$$

ch right-hand member the possible and impossible parts

may be separated, and the whole expression may be equ
to $P + Q\sqrt{-1}$; so that

$$P = f(y) - f''(y)\frac{z^2}{1.2} + f''''(y)\frac{z^4}{1.2.3.4} \ldots\ldots$$

$$Q = f'(y)z - f'''(y)\frac{z^3}{1.2.3} + f'''''(y)\frac{z^5}{1.2.3.4.5} \ldots\ldots$$

where P and Q are functions of two *independent* variabl
and z; and differentiating, we have

$$\frac{dP}{dy} = \frac{dQ}{dz}, \quad \frac{dP}{dz} = -\frac{dQ}{dy};$$

$$\frac{d^2P}{dy^2} = \frac{d^2Q}{dydz} = -\frac{d^2P}{dz^2}, \quad \frac{d^2Q}{dy^2} = -\frac{d^2P}{dydz} = -\frac{d^2Q}{dz^2}$$

hence, $\frac{d^2P}{dy^2}$ and $\frac{d^2P}{dz^2}$ being necessarily of different signs,
conditions of Art. 135 and 136 cannot be satisfied, and P
not admit of a maximum or minimum value; similarly al
is plain that Q does not admit of a maximum or minim
hence $f(y + z\sqrt{-1})$ is not a function admitting of a maxi
or minimum.

Secondly, let $y + z\sqrt{-1}$ be replaced by $r(\cos\theta + \sqrt{-1}\,s$
and $f(y + z\sqrt{-1})$ by $R(\cos T + \sqrt{-1}\sin T)$, r and R b
severally the moduli of $y + z\sqrt{-1}$ and of $f(y + z\sqrt{\cdot}$
and θ and T being real circular arcs; then, if it can be sl
that *some* value of θ is such as to render $R = 0$, y and z
such as to make

$$f(y + z\sqrt{-1}) = 0,$$

and it is proved that the equation $f(x) = 0$ has a root.

Since $\quad f(y + z\sqrt{-1}) = R(\cos T + \sqrt{-1}\sin T),$

$\therefore \quad f(y - z\sqrt{-1}) = R(\cos T - \sqrt{-1}\sin T);$

whence $\quad R^2 = f(y + z\sqrt{-1}) \times f(y - z\sqrt{-1}).$

Now to consider the most general case, suppose that al

derived-functions of $f(x)$ up to the mth exclusively vanish, when $x = y + z\sqrt{-1}$, viz. that

$$f'(y+z\sqrt{-1}) = 0,\ f''(y+z\sqrt{-1}) = 0, \ldots f^{m-1}(y+z\sqrt{-1}) = 0;$$

whence also,

$$f'(y-z\sqrt{-1}) = 0,\ f''(y-z\sqrt{-1}) = 0, \ldots f^{m-1}(y-z\sqrt{-1}) = 0.$$

Then, by means of Leibnitz's Theorem, see Art. 53, differentiating equation (10), and neglecting terms which vanish, we have

$$\begin{aligned} d^m.\mathrm{R}^2 &= f^m(y+z\sqrt{-1})\,(dy+dz\sqrt{-1})^m f(y-z\sqrt{-1}) \\ &\quad + f^m(y-z\sqrt{-1})\,(dy-dz\sqrt{-1})^m f(y+z\sqrt{-1}). \end{aligned}$$

Let R_m and ρ be the moduli severally of $f^m(y+z\sqrt{-1})$ and of $dx + \sqrt{-1}\,dz$, and let T_m and τ be the corresponding arcs, so that

$$\left.\begin{aligned} f^m(y+z\sqrt{-1}) &= \mathrm{R}_m(\cos\mathrm{T}_m + \sqrt{-1}\sin\mathrm{T}_m), \\ f^m(y-z\sqrt{-1}) &= \mathrm{R}_m(\cos\mathrm{T}_m - \sqrt{-1}\sin\mathrm{T}_m); \end{aligned}\right\} \tag{11}$$

$$dy \pm dz\sqrt{-1} = \rho(\cos\tau \pm \sqrt{-1}\sin\tau); \tag{12}$$

$$\therefore\quad (dy \pm dz\sqrt{-1})^m = \rho^m(\cos m\tau \pm \sqrt{-1}\sin m\tau); \tag{13}$$

making which substitutions

$$d^m.\mathrm{R}^2 = 2\mathrm{R}\mathrm{R}_m\rho^m\cos(\mathrm{T}_m - \mathrm{T} + m\tau). \tag{14}$$

Now as $f(y \pm z\sqrt{-1})$ does not admit of an absolute maximum or minimum value, that is, as it may have all or any values intermediate to $+\infty$ and $-\infty$, so R^2 (being of the form it is, viz. the square of R) does not admit of a maximum value; but it must have a minimum value; that derived-function of it therefore, which is the first not to vanish, must be consistent with such a singular value. Now on examining equation (14) τ is arbitrary, and $d^m.\mathrm{R}^2$ will evidently change sign, when τ, and therefore when dy and dz, receive certain values; as for instance, change τ to $\tau + \frac{2k+1}{m}\pi$, the sign of $d^m.\mathrm{R}^2$ will be changed; and such a change is inconsistent with R^2 being a minimum. The apparent difficulty however is obviated if one of the other factors of $d^m.\mathrm{R}^2$ vanishes; but R_m and ρ^m do not vanish, and therefore it follows that the only condition which is consistent

with z^2 being a minimum value is, that $z = 0$; this therefore is the minimum value; whence, by means of (8),

$$f(y + z\sqrt{-1}) = 0. \tag{15}$$

and therefore there always are some values of y and z such that equation (15) is satisfied; and therefore every equation of the form (1) has a root.

If in the general form of the root $z = 0$, the root is a possible one; but if z has a finite value, the root is impossible.

147.] Taking a to be the *general* symbol for the root of an equation, we may thus prove that the equation is divisible by $x - a$ without a remainder.

Let $f(x) = 0$ be the equation; then, since a is a root, $f(a) = 0$.

Observing now that Taylor's Series does not fail for a function of x, such as we have assumed $f(x)$ to be, and that the $(n+1)$th derived-function vanishes, because $f(x)$ is algebraical and of n dimensions, by means of equation (14) Art. 119, we have

$$
\begin{aligned}
f(x) &= f(a + x - a), \\
&= f(a) + f'(a)\frac{x-a}{1} + f''(a)\frac{(x-a)^2}{1.2} + \ldots\ldots \\
&\qquad\qquad + f^n(a)\frac{(x-a)^n}{1.2.3\ldots n};
\end{aligned}
\tag{16}
$$

the second member of which equation is, as $f(a) = 0$, divisible by $(x-a)$.

Hence also conversely, if $x - a$ is a factor of $f(x)$, $f(a) = 0$, or a is a root of the equation.

Hence we conclude that if any function of the algebraical form assumed in equation (1), Art. 145, vanishes for a particular value a of the variable, the function has a factor of the form $x - a$, and it is owing to *its* vanishing that the function vanishes; compare Art. 101.

148.] Hence also it follows, that every equation has as many roots as it has dimensions, and no more.

For dividing $f(x)$ in equation (16) by $(x - a)$, the highest power of x that remains is x^{n-1}, being that which is involved in the last term of it, viz. in $(x-a)^n$, whence there results an

expression of $n-1$ dimensions; which again, by virtue of Art. 146, has a root, and therefore is again divisible by a factor of the form $x-a$; whereby the expression is depressed to one of $n-2$ dimensions; and if a similar process be continued for $n-1$ times, we shall finally have an expression of one dimension, which will give the last root, and thereby the equation will have been resolved into n factors.

Thus suppose the n roots to be $a_1, a_2 \ldots\ldots a_n$, then

$$f(x) = (x-a_1)(x-a_2) \ldots\ldots (x-a_n). \tag{17}$$

Also it is manifest that no other value than one of the n roots can, when substituted for x, make any simple factor, and thereby the whole expression, to vanish; and therefore $f(x)$ has only n roots, some of which however may be equal; and therefore although all the n roots may not be different, yet there can never be fewer than n simple factors.

Again, if the coefficients of the several powers of x in $f(x)$ are real, and $f(x)$ has impossible roots, they must enter in pairs: so that, if a_i and a_j are two impossible roots which are conjugate to each other,

$$a_i = a + \beta\sqrt{-1} = \rho\{\cos\theta + \sqrt{-1}\sin\theta\},$$

$$a_j = a - \beta\sqrt{-1} = \rho\{\cos\theta - \sqrt{-1}\sin\theta\};$$

$$\therefore \quad (x-a_i)(x-a_j) = (x-a)^2 + \beta^2 = x^2 - 2\rho x\cos\theta + \rho^2,$$

which quadratic expression is essentially positive; and by a similar composition of other conjugate factors we have

$$f(x) = (x-a_1)(x-a_2)\ldots\{(x-a_1)^2+\beta_1^2\}\{(x-a_2)^2+\beta_2^2\}\ldots \tag{18}$$

and therefore $f(x)$ is the product of factors, simple or quadratic*.

149.] On the algebraical relation of $f(x)$ to its derived-function.

Let $f(x)$ be a function of the form (1), Art. 145, which has

* A further inquiry into the possibility and nature of the roots of equations would be out of place in the present Treatise, both because it does not so directly require Infinitesimal Calculus as to be cited in illustration of it, and because it is not elementary enough for our purpose. I would however recommend the advanced student to read "Cours d'Algèbre Supérieure," par J. A. Serret, Paris, 1849; and "Ouvres d'Abel, Christiania, 1839."

all its roots real, and therefore all its coefficients real quantities; and let the roots of $f(x)$ be $a_1, a_2 \ldots\ldots a_n$, so that

$$f(x) = x^n - p_1 x^{n-1} + p_2 x^{n-2} - \ldots\ldots \pm p_{n-1} x \mp p_n, \quad (19)$$

$$= (x-a_1)(x-a_2)\ldots\ldots(x-a_n), \quad (20)$$

the upper or lower sign being taken in (19) according as n is odd or even, and the roots being arranged in order of magnitude, viz.:

$$a_1 > a_2 > a_3 > \ldots\ldots > a_n; \quad (21)$$

$$\therefore \; f'(x) = nx^{n-1} - (n-1)p_1 x^{n-2} + \ldots\ldots \pm p_{n-1},$$

$$= (x-a_2)(x-a_3)\ldots\ldots(x-a_n) + (x-a_1)(x-a_3)\ldots\ldots(x-a_n)$$
$$+ (x-a_1)(x-a_2)\ldots\ldots(x-a_n) + \ldots\ldots$$
$$+ (x-a_1)(x-a_2)\ldots\ldots(x-a_{n-1}). \quad (22)$$

Let $x = a_1$, then observing that all the parts, except the first, of the second member of (22) disappear, and that by virtue of the arrangement of the roots, as indicated by (21), every factor of the first part is positive, it follows that, if $x = a_1$, $f'(x)$ is positive; similarly, if $x = a_2$, $f'(x)$ is negative. There is therefore (Cor. II, Art. 95) some value of x between a_1 and a_2 which makes $f'(x)$ vanish; a root therefore of $f'(x)$ lies between a_1 and a_2. Similarly we have the following results:

if $\quad x = a_1 \quad f'(x)$ is positive,

$\quad x = a_2 \quad f'(x)$ is negative,

$\quad x = a_3 \quad f'(x)$ is positive,

. . . .

$\quad x = a_n \quad f'(x)$ is positive or negative, according as n is odd or even. Hence the roots of $f'(x)$ are real, and lie between the roots of $f(x)$.

Let the roots of $f'(x)$ be $\alpha_1, \alpha_2 \ldots\ldots \alpha_{n-1}$, arranged in order of descending magnitude, then they stand to the roots of $f(x)$ in the relation indicated in the following table:

$$a_1 \qquad a_2 \qquad a_3 \,.\,.\,.\,.\,.\,.\,.\, a_{n-1}, \qquad a_n,$$
$$\alpha_1 \qquad \alpha_2 \qquad \alpha_3 \,.\,.\,.\,.\,.\,.\, \alpha_{n-1};$$

whence it appears, that the greatest root of $f(x)$ is greater than the greatest root of $f'(x)$, and the least root of $f(x)$ is less than

the least root of $f'(x)$. It is on account of this particular relation of the roots of $f'(x)$ to the roots of $f(x)$ that $f'(x)$ is sometimes called the *limiting equation* of $f(x)$.

Hence also it follows, that if all the roots of an equation be real, all the roots of each of its successive derived-functions will be real also.

These results admit of the following geometrical interpretation: Let the curve represented in fig. 23 be that whose equation is $y = f(x)$. As $f(x)$ has n real roots, $f(x)$ and therefore $y = 0$ at n points corresponding to them; that is, if $\text{OA}_1 = a_1$, $\text{OA}_2 = a_2, \ldots\ldots$ the curve cuts the axis of x at $\text{A}_1, \text{A}_2 \ldots\ldots$, that is, in n points. As $f(x)$ only $= \infty$ when $x = \pm \infty$, the ordinate is finite for all values of x between a_1 and a_2, a_2 and $a_3 \ldots\ldots$; and by the last Chapter as $f'(x) = 0$, and changes sign when $x = \alpha_1, = \alpha_2, = \ldots\ldots$, $f(x)$ is a maximum or minimum corresponding to these roots of $f'(x)$, and therefore we have maxima or minima ordinates at points intermediate to A_1 and A_2, A_2 and $\text{A}_3, \ldots\ldots \text{A}_{n-1}$ and A_n; that is, $\text{OB}_1 = \alpha_1$, $\text{OB}_2 = \alpha_2 \ldots\ldots$. Also as n is odd or even will the curve towards the left, when $x = -\infty$, be below or above the axis of x.

150.] Hence it appears that, if $f(x)$ has m roots equal to each other, $f'(x)$ has $(m-1)$ roots equal to each of the equal roots of $f(x)$; for if $a_1 = a_2 = \ldots\ldots = a_m$, then $\alpha_1 = \alpha_2 \ldots\ldots = \alpha_{m-1}$, as the αs are intermediate to the as; which proposition is also thus manifest.

Let m roots of $f(x)$ be equal to one another and to a, and let Q_x symbolize the product of the other $n-m$ roots, then

$$f(x) = (x-a)^m \text{Q}_x;$$

$$\therefore \quad f'(x) = (x-a)^{m-1} \left\{ m\text{Q}_x + (x-a)\frac{d.\text{Q}_x}{dx} \right\},$$

and $f'(x)$ has $m-1$ roots equal to each of the m equal roots of $f(x)$.

Hence if $f(x)$ has equal roots, they may be determined by the method of finding the greatest common measure of $f(x)$ and $f'(x)$, and $f(x)$ may be depressed by as many dimensions as it has equal roots.

The latter proof of this proposition manifestly reaches the case of equal impossible roots which the former may not resolve.

151.] **Given an equation $f(x) = 0$, which has real coefficients; it is required to determine the number of real roots which it contains, and the limits of them.**

The following process, due to M. Sturm, and now generally known by the name of "Sturm's Theorem," theoretically completes the subject of synthetical expressions, and is one of the greatest modern discoveries in Algebraical Analysis.

First let us consider $f(x)$ to have been cleared of *equal* factors by means of the last Article, so that $f(x)$ has not roots equal to each other; then we may enuntiate the following Theorem:

THEOREM.—Let $f(x)$ be a function of x of real coefficients, of which $f'(x)$ is the derived-function; let $f(x)$ be divided by $f'(x)$ in the way of finding the greatest common measure, but with the peculiarity of the sign of a remainder always being changed before it be made a divisor, and let this process of division be continued until it terminates by giving a remainder independent of x and not vanishing; which is always the case when $f(x)$ has no equal factors; and let these successive remainders thus modified be symbolized by

$$f_1(x),\ f_2(x) \ldots\ldots f_r(x),$$

so that we have the following system of equations:

$$\left.\begin{aligned} f(x) &= f'(x)\, q_1 - f_1(x), \\ f'(x) &= f_1(x)\, q_2 - f_2(x), \\ f_1(x) &= f_2(x)\, q_3 - f_3(x), \\ &\ldots\ldots\ldots\ldots \\ f_{n-3}(x) &= f_{n-2}(x)\, q_{n-1} - f_{n-1}(x); \end{aligned}\right\} \qquad (23)$$

$f_{n-1}(x)$ being the last factor and independent of x; and not vanishing because by hypothesis $f(x)$ has no equal roots.

Let a and β be two numbers of which a is the less (regard being had to its sign); substitute a for x in the series

$$f(x),\quad f'(x),\quad f_1(x) \ldots\ldots f_{n-1}(x), \qquad (24)$$

and write down in the same order the signs of the results; and count the number of sequences of two terms having contrary signs in this series of results; and suppose that A is that number.

Substitute β in the same series of functions, and count

as before the number of sequences of two terms with contrary signs, and suppose it to be equal to B,

Then there are A—B roots of $f(x)$ lying between a and β; that is, the number of the excess of variations of signs in successive terms, when a is substituted for x, over that when β is substituted for x, is the number of real roots of $f(x)$ greater than a and less than β.

For suppose a to be a real root of $f(x)$; and in the series (24) for x let there be substituted $a-h$; then the series of functions becomes by equation (16), Art. 99,

$$f(a) - f'(a)\frac{h}{1} + f''(a+\theta h)\frac{h^2}{1.2},$$

$$f'(a) - f''(a+\theta h)\,h,$$

$$f_1(a) - f_1'(a+\theta h)\,h,$$

$$f_2(a) - f_2'(a+\theta h)\,h,$$

. . . .

in each of which we may take h such an infinitesimal, that the terms involving it must be neglected when added to a finite quantity; then, since a is a root of $f(x)$, and therefore $f(a) = 0$, the signs of the series of terms are the same as those of

$$-f'(a),\quad f'(a),\quad f_1(a)\ldots\ldots \tag{25}$$

Similarly let $a+h$ be substituted for x, then the signs of the series of results are the same as those of

$$f'(a),\quad f'(a),\quad f_1(a),\quad f_2(a)\ldots\ldots \tag{26}$$

whereas then of (25) the first two terms are affected with opposite signs, the first two terms of (26) have the same signs; therefore by making x increase from a quantity a little below a real root to a quantity a little above it, a variation of sign in the series of functions (24) is lost; that is, what was a sequence of opposite signs has become a sequence of the same signs.

Also a similar loss of variation of signs takes place whenever x passes through a root; and therefore, if we make x to grow by infinitesimal increments from a to β, every time that its value becomes that of a real root of the equation the series of signs of $f(x)$, $f'(x)$, $f_1(x)$ ...... loses a variation, and only then; hence there are as many real roots between a and β as there are more variations for a than for β.

In calculating the successive values of the series of functions (24) we may observe that any function may be multiplied or divided by a positive, but not by a negative quantity, as the sign would thereby be changed, and it is from the signs that the proposition is deduced.

Ex. 1. $$x^3 - x^2 - 4x + 3 = 0 = f(x),$$

$$3x^2 - 2x - 4 \quad = f'(x),$$

$$\begin{array}{r} 3x^2-2x-4)\ 3x^3 - 3x^2 - 12x + 9\ (x-1 \\ \underline{3x^3 - 2x^2 - 4x} \\ - x^2 - 8x + 9; \end{array}$$

or
$$\begin{array}{r} -3x^2 - 24x + 27 \\ \underline{-3x^2 + 2x + 4} \\ -26x + 23; \end{array}$$

$$\therefore\ f_1(x) = 26x - 23.$$

$$\begin{array}{r} 26x-23)\ 78x^2 - 52x - 104\ (3x+17 \\ \underline{78x^2 - 69x} \\ 17x - 104 \\ \\ 442x - 2704 \\ \underline{442x - 391} \\ -2313; \end{array}$$

$$\therefore\ f(x) = x^3 - x^2 - 4x + 3,$$

$$f'(x) = 3x^2 - 2x - 4,$$

$$f_1(x) = 26x - 23,$$

$$f_2(x) = 2313.$$

The signs of which series of functions, corresponding to the following values, are

when	$x = -2,$	$-,\ +,\ -,\ +,$
	$x = -1,$	$+,\ +,\ -,\ +,$
	$x = 0,$	$+,\ -,\ -,\ +,$
	$x = 1,$	$-,\ -,\ +,\ +,$
	$x = 2,$	$-,\ +,\ +,\ +,$
	$x = 3,$	$+,\ +,\ +,\ +.$

Hence as a change of sign is lost in passing from $x = -2$ o $x = -1$, a root lies between -2 and -1; and as changes are gain lost in passing from $x = 0$ to $x = 1$, and from $x = 2$ o $x = 3$, two other roots lie severally between 0 and 1, and etween 2 and 3. Hence the equation has three real roots, he positions of which have been determined.

152.] We have supposed that none of the quantities $f'(x)$, $_1(x)$, $f_2(x) \ldots\ldots f_{n-1}(x)$ vanishes, when for x we substitute a; ut if one does vanish, the number of variations of signs is not ltered by it. For suppose $f_i(a) = 0$, then, since

$$f_{i-1}(a) = q_{i+1}\, f_i(a) - f_{i+1}(a),$$

re have
$$f_{i-1}(a) = -f_{i+1}(a),$$

ieither of which can vanish; for if *two* consecutive functions anish when $x = a$, then tracing backwards by means of the quations of the group (23) we should have $f_{i-2}(a) = 0 \ldots\ldots$ $_1(a) = 0$, $f'(a) = 0$, $f(a) = 0$: which last two values can nly consist when $f(x)$ has equal roots, and this is contrary to he supposition made at first. Hence $f_{i-1}(a)$ and $f_{i+1}(a)$ are of ontrary signs and form a variation; and therefore of the three unctions $f_{i-1}(a \pm h)$, $f_i(a \pm h)$, $f_{i+1}(a \pm h)$, there must be either permanence and then a variation, or first a variation and then permanence. Hence the vanishing of $f_i(a)$ does not affect he truth of the theorem.

153.] Cor. to preceding Theorem.

Hence follows an easy method of determining the whole umber of real and impossible roots of an expression of the orm $f(x) = 0$.

Whatever be the number and value of the real roots, they nust be between $-\infty$ and $+\infty$; let us therefore form as bove the series of functions

$$f'(x), \quad f_1(x), \quad f_2(x) \ldots\ldots f_{n-1}(x),$$

he number of which is generally $(n+1)$, $f(x)$ being of n dimenions; and let $+\infty$ or a very large quantity be substituted for x, so that the signs of the functions are the same as the signs f the first terms; let m be the number of variations of signs f consecutive terms of this series.

And now let $-\infty$ be substituted for x, then the signs of the

functions of even dimensions will be the same as before, the signs of those of odd dimensions will be the contrary.

There will therefore in the latter substitution be as ma variations of succeeding terms as there were permanences the former, that is, there will be $n-m$ variations.

And as all the real roots are comprised within these lim their number (by the Proposition, Art. 151) must be $n-m$- or $n-2m$, and therefore the number of impossible roots is 2

There exist therefore as many pairs of impossible roots there are variations in the signs of the first terms of the fu tions $f(x)$, $f'(x)$, $f_1(x) \ldots\ldots f_{n-1}(x)$.

Ex. 1.
$$f(x) = x^3 - 3px + 2q,$$
$$f'(x) = 3x^2 - 3p,$$
$$f_1(x) = px - q,$$
$$f_2(x) = p^3 - q^2.$$

The series of signs of the first terms are the same as those

$$1, \quad 1, \quad p, \quad p^3 - q^2.$$

If therefore p be negative there is one variation, and the fore only one real root; and if p be positive there is one r root when p^3 is less than q^2, and if p^3 is greater than q^2 all t roots are real.

154.] Fourier's Theorem.

The following process was arranged by Fourier to separ the real and impossible roots of an equation; but as it o indicates a number which the sought number of real ro does not exceed, the discovery of M. Sturm renders it alm useless; however, as it advantageously exhibits the relati between the successive derived-functions in an algebraical po of view, it is right to insert it in a treatise on Infinitesin Calculus.

Let $f(x)$ be a function of x of the form (1), Art. 145, a which has been cleared of equal factors, and suppose all coefficients to be real; let the several derived-functions of be formed, whereby we have a series

$$f(x), \quad f'(x), \quad f''(x) \ldots\ldots f^n(x); \qquad (2$$

and let α and β be two numbers of which α is the less, th there cannot be more real roots of $f(x)$ between α and β th

the excess of the number of alterations of sign in the above series of functions, when α is substituted for x, over the number resulting from the substitution of β.

For suppose a to be root of the equation $f(x)$, then, since $f(a) = 0$, by the substitution of $a-h$ for x in the above functions, the series becomes, when h is infinitesimal,

$$-f'(a)\,h,\quad f'(a)\,\ldots\ldots f^n(a);$$

and when $a+h$ is substituted for x,

$$f'(a)\,h,\quad f'(a)\,\ldots\ldots f^n(a);$$

and thus, supposing none of the derived-functions to vanish, the passage of the substituted quantity, from a value a little below a real root to one a little above it, causes a variation of sign to be exchanged for a permanence, and *pari ratione* as such a variation will be lost whenever the substituted quantity passes through a root, it follows that, as many real roots as there are lying between α and β, so many losses of variations of signs at least will there be in the series of functions above, when we pass gradually from α to β.

At least, I say; for the series of signs may also be affected by the vanishing of any of the subsequently derived-functions; for suppose b to be such as, when substituted for x, to cause $f^r(x)$ to vanish, and h to be an infinitesimal, then, if $b-h$ be substituted for x, we have for $f^{r-1}(x)$, $f^r(x)$, $f^{r+1}(x)$,

$$f^{r-1}(b),\quad -f^{r+1}(b)\,h,\quad f^{r+1}(b); \tag{28}$$

and when $b+h$ is substituted for x,

$$f^{r-1}(b),\quad f^{r+1}(b)\,h,\quad f^{r+1}(b). \tag{29}$$

If $f^{r-1}(b)$ and $f^{r+1}(b)$ (viz. the first and last terms of (28) and (29)) are of contrary signs, then we shall have a variation and a continuance both in (28) and (29) so that no change will be lost. But if $f^{r-1}(b)$, $f^{r+1}(b)$ are of the same sign, then in (28) we shall have two variations, and in (29) two continuations, so that *two* changes of sign will be lost.

Similarly it may be shewn, that if many successive derived-functions vanish for a particular value of x, an even number of variations of sign may disappear.

There may therefore be losses of variation of sign in the series of functions given in (27) at other values of x than roots of $f(x)$, but variations must nevertheless be exchanged for continuations at the roots; therefore the Theorem gives only a number which is *not less* than the number of real roots. The advantage of Sturm's Theorem is, that it gives the exact number of real and of impossible roots.

155.] The rule commonly known by the name of Des Cartes' Rule of Signs is a particular case of Fourier's Theorem; viz. that in the general equation $f(x) = 0$, the number of positive roots cannot exceed the number of variations of sign of the successive terms, and the negative roots the number of continuations of signs.

Firstly, let $x = 0$ in the series of functions (27), then the signs of $f(x)$, $f'(x)$, ... $f^n(x)$ are the same as those of the several and successive terms of $f(x)$ taken from right to left; and when $x = \infty$, the signs of the functions are all positive. Hence there can be no more real positive roots than there are changes of sign in the successive terms of $f(x)$.

Secondly, let $x = -\infty$, then the series of functions form only variations of sign, of which there are of course n, and therefore the number exceeds the number of variations, when $x = 0$, by the number of permanences in the terms of the equation. Hence the number of negative roots cannot exceed the number of continuations of signs.

156.] Taylor's Series also furnishes a method, which was invented by Newton, for finding a number greater than the greatest root of an equation.

Let $f(x) = 0$ be an equation of n dimensions of the form (1) Art. 145, and for x let us substitute $y + h$;

$$\therefore \quad f(y+h) = f(h) + f'(h)\frac{y}{1} + f''(h)\frac{y^2}{1.2} + \dots + f^n(h)\frac{y^n}{1.2.3\dots n}.$$

Suppose that such a value is given to h as to make $f(h)$, $f'(h)$ ...... $f^n(h)$ all positive, then by Fourier's Theorem no root of the equation can lie between h and $+\infty$; therefore h is greater than the greatest positive root.

157.] Taylor's Series is also useful for approximating to a root of an equation.

Suppose two values a and β, the difference between which is small, to have been found (of which a is the less), which, when substituted for x, give results with different signs, then a root of the equation lies between them. To determine it, let us suppose the root to be $a+h$; then $f(a+h) = 0$; but by Art. 101, equation (21),

$$f(a+h) = f(a) + f'(a+\theta h)\, h,$$

$$\therefore \quad h = -\frac{f(a)}{f'(a+\theta h)};$$

and neglecting θh when added to a, since h is small, we have

$$h = -\frac{f(a)}{f'(a)};$$

in which case however $f'(a)$ ought to be large in comparison of $f(a)$, otherwise the result is inconsistent with our supposition of h being small.

DIFFERENTIAL CALCULUS.

PART II.

GEOMETRICAL APPLICATIONS.

CHAPTER IX.

ON GEOMETRY.

SECTION 1.—*On the adjustment of the Principles of Geometry and Infinitesimal Calculus.*

158.] IT will by this time have become tolerably plain to the attentive reader, that the characteristic property of Number, which is the foundation of Infinitesimal Calculus, is that of continuous and infinitesimal growth; and that Differentiation is the mathematical expression of the Law of Continuity. Now our object in the following pages is to apply the propositions which have been proved above to questions of pure geometry; and therefore it is necessary so to modify or enlarge the principles of that science, as to adjust them to those of Infinitesimal Calculus.

As it is not however our intention to write a treatise on the principles or difficulties of Elementary Geometry, we shall rather enuntiate axioms and definitions, and state results, than prove propositions, leaving the last to be effected by our applications; neither shall we discuss the methods by which we have arrived at them, in the belief that a rational understanding of the first Chapter of the present Treatise is sufficient to explain them. In accordance with illustrations therein given, we have introduced the ideas of *motion* and of *limits;* motion perhaps as having to do with the generation of geometrical quantities, but

chiefly as involving the property of infinitesimal divisibility, which is necessary to a due conception of the latter property of *limits*; motion however, as we have introduced it, does not encroach on the subject of mechanics, wherein we treat of motion as the effect of certain causes, and discuss its circumstances, as e. g. the particular law of force which produces it, the velocity with which the moving material changes position, which necessarily involves time, and so on: but in what follows we consider motion as a simple act, a primary conception as a quality of matter; and if it tends, as it does, to give clearness to our first geometrical conceptions, it is nothing but a servile adherence to an inferior, though customary method, which would hinder us from introducing it.

It is conceived that all geometrical quantity, whether linear, superficial, or spatial, is from its very nature capable of increase or decrease to an infinite extent. A line may be very long, nay of an infinite length, or very short; space may be very small, such as, so to speak, it would require a microscope of almost infinite power to render visible, or it may be very large. Whenever such quantities vary, they do so in accordance with the law of continuity; they cannot pass from one magnitude to another without passing through all intermediate magnitudes; they grow larger and larger, or less and less. This capability of increase or decrease is involved in our idea of geometrical quantity; it is necessary to its completion; and if it be omitted, our notions fall short of the properties of the subject-matter of the science.

There are however limits within which this variation is included; the superior limit of geometrical magnitude of the concrete kind called *space* is infinite space: so of superficial and linear magnitudes, the superior limits are respectively infinite superficies, and a line of infinite length.

The inferior limit of all these is the same, the geometrical zero, a *point*.

159.] Of the definitions of geometrical quantities founded on such notions, the following are useful for our present object.

I. A *point* is the inferior limit of geometrical space.

II. A *sphere* is the locus of a point of space, which is always at the same distance from a given point.

III. A *plane* is the surface of a sphere, the radius of which is infinitely great.

IV. A *circle* is the locus of a point, which is always at the same distance from a given point, all the points being in one plane.

V. A *straight line* is the arc of a circle, the radius of which is infinitely great.

VI. A *triangle* is a plane figure contained by three straight lines meeting one another, two and two.

VII. And if the triangle be isosceles, the sides of that triangle, having a finite base and the vertex at an infinite distance, are *parallel straight lines*.

As this is not intended to be an accurate treatise on the principles of geometry, many words are used which have not been defined, as line, locus, &c.; these however are to be taken in their ordinary significations, and it is to be observed, with respect to these definitions and conceptions, that the surfaces, lines, &c. they refer to, are only approximations to the accurate ones. But they are such approximations as may differ from the real ones by quantities as small as we please; and as these small quantities may be infinitesimals, such that it would require an infinity of them to make a finite quantity, and we do not take an infinite number of them, these differences must, in conformity with what has been said in the first Chapter, be neglected, and our definitions are *practically* rigorously exact.

Having defined a plane, as we have done, to be the limiting spherical surface when the radius becomes infinitely great, it follows that the extreme positive side of the plane, when continued, runs into the extreme negative side; that is, having traced the plane as far as we can on the positive side, we meet it again on the negative; and although the surface appears to be discontinuous, it is not in reality so: the positive side being continued into the negative, and the apparent discontinuity arises from the defect in our power of apprehending and symbolizing such quantities. Thus then, if we have any continuous curve traced on the plane, and the curve runs off to the extreme positive side of the plane, we ought not to consider it to stop or to have points of discontinuity, but we must consider the branches of it to be continued, and must look for them on the

negative side of the plane. We may borrow from the figure of the earth, and our mode of determining position on its surface, an illustration of what is here intended. We measure, say from the meridian of Greenwich, degrees along the equator to 180° east longitude; and then, instead of proceeding further on and measuring in the same direction, we measure backwards, and reckon degrees of longitude west: and what would be 181° east longitude becomes 179° west. If then east corresponds to the positive direction, west does to the negative.

It is worth remarking how exactly our ideas of a plane coincide with the definition I have given. We speak of the surface of water as a plane, and consider it to be *level*, whereas it is a portion of the surface of a sphere, whose radius is very large compared with the area we take, say, 4000 miles compared with a few inches.

So again as to our conception of a straight line. A straight line being a particular instance of a circle, is a continuous line; it does not terminate at positive infinity nor at negative infinity, but the two branches of the line are connected with one another, running, if we may so speak, round the circle of which the radius is infinity, and joining together. If then we take any given point on the circle as the origin, the distance to the opposite extremity of the diameter of the circle is positive infinity, and we do not measure or follow the line further in this direction, but considering the line to be continued beyond that point, we meet it on the opposite side, and measure it backwards. There is no point of discontinuity in the line: the line proceeds in the same direction; it has been positive infinity; the pole or extremity of the diameter of the circle has been passed, and then the line becomes negative infinity. The illustration above given from the figure of the earth aptly illustrates our meaning in this case. Considering any meridian to be the very large circle, and taking any place on it to be the origin, the "antipodes" to it becomes either positive infinity or negative infinity, according as we measure in the positive or negative direction; the sign of the quantity changes immediately after the pole has been passed, and what was positive infinity becomes negative infinity. Therefore in this point of view infinity is not a quantity incapable of increase, for the line may be continued round and round the meridianal circle as often as we please; there is

no limit to the quantity: the limit is to our powers of symbolising such quantities.

It is worth observing too, that the definition given of parallel straight lines enables us to avoid the difficulty connected with our first introduction to the theory and properties of such lines. Having shewn, as Playfair has done, that the exterior angles of a triangle are together equal to four right angles, it follows that the interior angles are equal to two right angles: but if the base of a triangle remains finite, and the vertex is removed further and further, the vertical angle becomes less and less, and diminishes without limit, in which case the sum of the base angles is equal to two right angles, and the sides become parallel straight lines, whence the properties of such lines, which are enuntiated in the xxixth proposition of the first book of Euclid, immediately follow.

A good illustration of this theory occurs in the Phenomena of Parallax. If the angles subtended at the centre of the earth by the sun and any fixed star, whose parallax has not been discovered, be observed when the earth is in perihelion and at aphelion, it is found that, notwithstanding the extreme delicacy of our instruments, the sum of these two angles is exactly equal to two right angles. Taking then the two positions of the earth to be the extremities of the base of a triangle, and the line passing through the sun's centre and terminated by them to be the base, and the fixed star to be the vertex, it appears that, although the base of the triangle be 190,000,000 of miles, the angle subtended by it at the vertex is too small to be measured, and the two lines drawn to the star from the earth, at the two positions of it, are to all appearance parallel straight lines.

160.] In corroboration also of what has here been stated, the following are a few out of a great many striking instances:

In differentiating $\tan\theta$, we have

$$\frac{d.\tan\theta}{d\theta} = (\sec\theta)^2,$$

which is necessarily a positive quantity; and therefore by Theorem I, Art. 95, θ and $\tan\theta$ are always increasing and decreasing simultaneously, and therefore as θ increases $\tan\theta$ increases. Now as θ approaches to $90°$, $\tan\theta$ becomes $+\infty$,

and immediately after θ has passed 90°, $\tan\theta$ becomes $-\infty$, indicating that negative infinity is positive infinity increased; that is, as θ has increased and passed through 90°, $\tan\theta$ has increased from $+\infty$ to $-\infty$. And so again, as θ increases from 90° to 180°, $\tan\theta$ is continually increasing from $-\infty$ to 0, and passes through 0, and increases to $+\infty$, which is the value of $\tan\theta$ when $\theta = 270°$; and so on, as θ increases, $\tan\theta$ is continually increasing, travelling, if we may so say, round the circle of which the straight line along which $\tan\theta$ lies is conceived to be the limit when the radius of the circle is infinitely great. It is impossible not to remark how exactly this illustration agrees with what has been said in Chapter I on the order of infinitesimals. For corresponding to every 180° through which θ turns, $\tan\theta$ passes from 0 to ∞, and on through ∞ to 0 again; that is, the path through which $\tan\theta$ has travelled is infinite, although θ has passed over only a finite angle; and therefore, when θ has revolved through 360° and 540° and 720°, and so on, $\tan\theta$ has travelled over a length of line equal to twice, three times, four times, &c., the infinite length corresponding to a revolution of θ through 180°; and thus we have infinities bearing a finite ratio to each other. Conceive moreover θ to have revolved an infinite number of times through 180°, then the distance over which $\tan\theta$ will have travelled will be an infinity of infinities, that is, will be $(\text{infinity})^2$; and thus we obtain different orders of infinity.

Again, suppose we have given the following problem: To find the Maximum and Minimum values of y, when

$$y = \frac{(x+2)^2}{(x-3)^3},$$

$$\frac{dy}{dx} = -\frac{(x+2)\,(x+12)}{(x-3)^4},$$

$y = \infty$, when $x = 3$, but as $\frac{dy}{dx}$ does not change its sign, this value of y is neither a maximum nor a minimum. How then is the result to be interpreted? As follows: Since $\frac{dy}{dx}$ is negative, y decreases as x increases, and when x is a little less than 3, $y = -\infty$: but when x is a little greater than 3,

$y = +\infty$; therefore, as x has passed through 3, the value of y has changed from $-\infty$ to $+\infty$, but y has decreased during this progressive increase of x, therefore $+\infty$ is $-\infty$ decreased; therefore y has not reached a minimum or a maximum value when $x = 3$, because it has not become $-\infty$, and then returned, but it has gone on decreasing. And if we draw a graphical representation of the curve corresponding to the equation, such as that in fig. 24, the phenomena explain themselves. The curve on the negative side of the axis of y is of the form CB, where OB $= 2$; and if OA $= 3$, the curve is continually approaching the line drawn through A parallel to the axis of y, and when x is nearly 3, y is $-\infty$: but when x is greater than 3, y is $+\infty$; that is, the curve has crossed the asymptote at the pole of the circle of infinite radius opposite to A, and has returned in the direction EF, the branch in the direction of E being a continuation of that in the direction of D. Similarly the branch in the direction F would, if produced, unite itself to that in the direction C, having crossed the axis of x at the pole opposite to O.

In corroboration of this theory, it will appear that if the criteria, which will be discussed in the next Chapter, be applied, whenever a curve is of the form fig. 24, at points such as those where the branch E meets the branch D, and crosses the asymptote, we have all the characteristics of a point of inflexion; and if the curve be such as in fig. 25, we have the characteristics of a point of *embrassement*; and whenever such as is represented in fig. 26, all the conditions of a maximum ordinate.

And so again whenever a branch of a curve continues to infinity, it always returns in some way or another; and in whatever manner a rectilineal asymptote be drawn, no branch of the curve ever goes off asymptotic to it without returning in one of the ways indicated in the figures 24, 25, 26; and it seems impossible to account for such phenomena except on the theory explained above, viz. that the plane and the straight line are respectively the superior limits of the sphere and the circle, when the radii become infinitely large *.

* For a further elucidation of many points in elementary geometry more or less connected with the present subject, I would refer the reader to a small Treatise on the Difficulties of Elementary Geometry, by F. W. Newman, M.A., formerly Fellow of Balliol College, Oxford; Longman and Co., London, 1841.

SECTION 2.—*On the Extension of Symbols of Direction.*

161.] In algebraical geometry, and therefore in the applications of the Differential Calculus to the theory of plane curves, we meet with symbols of two distinct characters; symbols of quantity, such as $a, b, c, \ldots\ldots x, y, z, \theta, \phi, \psi \ldots\ldots$, when symbolical respectively of lines and angles: and symbols of *direction*, $+, -, +\sqrt{(-)}, -\sqrt{(-)}$, &c. Our object is so to enlarge our method of interpreting symbols of this second kind, as to comprehend those which are usually called Impossible, of which however we shall discuss only two, viz. $+\sqrt{-}, -\sqrt{-}$, or as they may be written, in accordance with the index law, $+(-)^{\frac{1}{2}}$, $-(-)^{\frac{1}{2}}$ *.

As to symbols of quantity, it is to be observed that, when we symbolize a line by a, we do not mean that a is the absolute length of the line, for all lengths can only be relative, and there must be some modulus or standard to compare them with: but we intend a line which is in length a times some arbitrary, though for the time fixed, standard unit. So a line symbolized by b is a line b times in length some unit. Thus then a, b are numerical quantities, not concrete magnitudes, but abstract quantuplicities, the subject-matter of arithmetical algebra, and therefore subject to its laws; they do not designate the absolute lengths of lines, but the number of times a certain concrete unit is to be taken. So again if an area be symbolized by ab; a and b are abstract numbers, which must be multiplied together by the laws of arithmetical algebra, and their product is the number of times the superficial unit is to be taken. Let it therefore be carefully borne in mind that this is the meaning of the several symbols of quantity, whether constant or variable, which we shall use in the following Chapters. Suppose then we have a line symbolized by a, and we fix upon a certain point as the origin from which lines are to be measured, any line drawn

* For a fuller explanation of the principles of explaining these and such like symbols, we would refer to Études Philosophiques sur la Science du Calcul, par M. F. Vallès, 8vo. Paris, 1841; and for the general theory of the meaning of $(+)^{\frac{p}{q}}$ to Dr. Peacock's Algebra, 8vo. Cambridge, vol. i. 1842, vol. ii. 1845; to Mr. Warren's Treatise on the Square Root of Negative Quantities, 8vo. Cambridge, 1828, and to many papers in the Cambridge Mathematical Journal.

from it, equal in length to a times the linear unit, will fulfil the requirements of the single symbol a. But inasmuch as an indefinite number of equal lines may be drawn from any one point, thus far we have no means of determining which of all such lines is intended; hence arises the necessity of some other symbols to indicate direction, or, as they are called, ***symbols of direction*** or ***affection***. One or two of the most simple cases of these we proceed to explain, feeling assured that the principle of explanation is so entirely in harmony with the usual meaning of + and −, that it ought not to be omitted in an elementary treatise; and also because it enables us to shew that an algebraical curve, though apparently discontinuous and confined within certain fixed limits, is not in reality so, but extends to infinity in all directions. Other parts of the theory (some of which are as yet not sufficiently established) we omit, as unsuited to our present object.

162.] Suppose o, fig. 27, to be the point from which lines are to be measured, and OA $= a$ times the linear unit to be drawn from o towards the right hand. Now since, as we said above, any line drawn from o, a times the linear unit in length, will be symbolized by a, it is necessary to fix on some *originating* direction; suppose this to be OA, and any line measured from o towards A to be affected with the symbol of direction +; if then, after a line has undergone any operation or a series of operations, it comes into the position OA, it is still to be symbolized by + : and, if the line be a, by $+ a$. Such an operation we might conceive to be a reciprocating one, the line at one time being in the position OA, and at another in the position O′A′, having moved sideways, and assumed all intermediate positions. Or we may conceive that the line OA (see fig. 28) has revolved round the point o, and, having turned in the plane of the paper through 360°, has again come into its original position, and so on continually; and it is manifest that as often as it has revolved through any multiple of 360°, it has assumed its original position OA, and is therefore to be symbolized by $+ a$. So also there are many conceivable ways in which the line may have moved, and that periodically, and at the end of a complete period be in the position OA. But have we any other customary mode of indicating direction, to serve as a guide which of these conceivable operations to take? We have.

Whenever a line equal in length to a is measured from O towards the left, we symbolize it by $-a$; if therefore either $(-)$ were a symbol for the operation of one oscillation having been performed on the line, i.e. the line having passed into the position O′A′ (see fig. 27): or $(-)$ symbolized the line OA (fig. 28), having been turned through 180°, either would account for the negative sign of affection, and $(-)$ would be the symbol of the operation; but under the first hypothesis, the line at one stage of the process will be half on the positive side of the origin and half on the negative; if therefore the operation be continuous, which it is, in passing from $+$ to $-$, there should be some symbol to indicate that particular stage; it does not however appear that we have any symbol of the kind; and such a motion, and a line in such a state, are what in our ordinary geometrical conceptions we do not use nor contemplate. Let us therefore consider whether we have not symbols to indicate a line in any intermediate position between OA and OA_1, conceiving the line to pass from the one position to the other by means of revolving through 180°.

As we said before, whenever the line is measured from O in the direction OA, it is to be affected with a $+$ sign. Taking therefore O as the origin of line, and OA as the *direction line* from which symbols and operations of affection are to be originated, whenever a line, as e.g. OA, has turned an integral number of times through 360°, it is to be affected with the sign with which it started. If therefore it was affected with the $+$ sign at first, indicating that it started from OA, and if $+$ be the symbol of turning through 360°, after one revolution the symbol of affection is $+$ on the back of $+$, i.e. according to the index law, $+^2$; similarly after two revolutions, $+^3$; and after $(n-1)$ revolutions, $+^n$. Supposing therefore that the line which is of the length a, when along the originating direction OA, is unaffected with any sign: $+a$ means that the line has turned through 360°, and has come again into the position whence it started; and so $+^n a$ means that a line of length a has revolved n times from the direction of origination, and is in the position OA; whence it appears (in accordance with the arithmetical meaning and law of $+$), that $+$ is, for symbolical purposes of direction, equivalent to $+^n$, n being a whole number.

In conformity then with the algebraical law of indices $+^{\frac{1}{2}}$ is

the symbol of that operation, which, being performed twice, one on the back of the other, brings the symbol into the value $+$; that is, if $+$ signifies turning the line through 360°, $(+)^{\frac{1}{2}}$ indicates turning it through 180°, but $-$ symbolizes this operation,

$$\therefore \quad +^{\frac{1}{2}} = -, \text{ and } (-)^2 = +;$$

or the operation symbolized by $(-)$ performed twice, one upon another, is equivalent to the operation signified by $+$, and means turning a line through 360°. Similarly again $(+)^{\frac{2n+1}{2}}$ is equivalent to $-$, for it is equivalent to $+^{n+\frac{1}{2}} = +^n(+)^{\frac{1}{2}} = +^n -$; and this coincides with the ambiguity we have always in the sign of $+^{\frac{1}{2}}$, for it may, as far as the form $+^{\frac{1}{2}}$ teaches, be either $+$ or $-$. If therefore the $+$, whose root has to be extracted, be raised to an even power, its root is to be affected with a positive sign; but if the $+$ be $+^{2n+1}$, then the square root is $+^n+^{\frac{1}{2}}$, which is equivalent to $-$, and the root must be affected with the negative sign. Hence also it is plain that $\sqrt{(-a)} \times \sqrt{(-a)}$, which equals $\sqrt{a^2}$, can only be $-a$, because the $+$, with which a^2 is affected under the radical, is of only the first power.

Therefore we have shewn that in symbolical geometry

$$\left.\begin{aligned} &1\text{st}, \quad +^n = +, \\ &2\text{d}, \quad +^{\frac{2n+1}{2}} = -. \end{aligned}\right\}$$

163.] So again $+^{\frac{1}{4}}$ symbolizes that operation which, being performed twice, one on the back of the other, is equivalent to $+^{\frac{1}{2}}$, i.e. to $(-)$, and, being performed four times successively one on the back of the other, is equivalent to $+$;

$$\therefore \quad +^{\frac{1}{4}} = (-)^{\frac{1}{2}};$$

and therefore, as $-$ indicates that a line is to be turned through 180°, so $(-)^{\frac{1}{2}}$ means that a line is to be turned through 90°. Whenever then a line is affected with $(-)^{\frac{1}{2}}$, which is equivalent to $+^{\frac{1}{4}}$, as its symbol of direction, that line is to be drawn at right angles to the original direction of origination, viz. in the direction OA_2 (see fig. 28); and whenever the symbol of direction is $+^{\frac{3}{4}}$, which $= +^{\frac{1}{2}}+^{\frac{1}{4}} = -(-)^{\frac{1}{2}}$, the line which is affected with it is to be drawn in the direction OA_3. Similarly $+^{\frac{4n+1}{4}}$ indicates a line drawn in the direction OA_2, and $+^{\frac{4n+3}{4}}$ a line drawn in the direction OA_3. So also

$+^{\frac{1}{n}}$ means that that line with which it is affected is to be drawn at an angle of $\frac{360^\circ}{n}$ to the originating direction OA.

164.] We have inserted the above method of explaining symbols of direction, which are usually termed Impossible and passed over in silence, because it is clearer to the perception than another method which has received copious elucidation from Dr. Peacock and Mr. Warren: that viz. in which $\cos\theta + \sqrt{-1}\sin\theta$ is considered as the symbol, and whereby, when it is affixed to a line, say ρ, the direction is indicated in which the line is to be drawn; thus

$$\rho(\cos\theta + \sqrt{-1}\sin\theta) \tag{1}$$

represents a line of length ρ drawn at an angle θ to the originating direction. The two methods coincide at those points which will be most useful in the sequel; thus let $\theta = 0$, then the line represented by (1) is ρ, and coincident with the zero operation, that is, with the line of origination; let $\theta = 90^\circ$, the line becomes $\rho\sqrt{-1}$, and is at right angles to the originating direction. Let $\theta = 180^\circ$, and the line is $-\rho$, that is, is the originating line produced backwards; let $\theta = 270^\circ$, and (1) becomes $-\sqrt{-1}\,\rho$, and is in a direction at right angles to and below the originating direction; and if $\theta = 360^\circ$ the line becomes $+\rho$, and lies in its original direction.

In accordance then with the interpretation of $\sqrt{-1}$ which such a symbol as (1) thus used involves, it will be observed that (1) correctly represents two sides of a *rectangle*; that is, fig. 29, if $\text{OP} = \rho$ and $\text{POM} = \theta$, $\text{OM} = \rho\cos\theta$ and $\text{PM} = \rho\sin\theta$, and as PM is affected with $\sqrt{-1}$, it is to be measured in a direction PM, which is perpendicular to OM; which lines therefore cannot be added (or subtracted), as they are not in the same line, but we may by an extension of interpretation suppose (1) to represent the diagonal OP of the parallelogram, of which OM and MP are the two containing sides.

It is also to be observed, that

$$\cos\theta + \sqrt{-1}\sin\theta = e^{\theta\sqrt{-1}}, \tag{2}$$

and that therefore $e^{\theta\sqrt{-1}}$ may be used as a symbol of direction; wherein θ expresses the angle of inclination to the originating line of the line which the symbol affects.

165.] To apply these principles to the delineation of plane curves from their equations, suppose $y = f(x)$ to be the equation to the curve; since x and y have already preoccupied the two directions at right angles to each other in the plane of the paper, which is (and conveniently so) called the *plane of reference*, we must seek for some other course by which a line, which has been measured in the positive direction, may be made to turn through 180° into the negative. Such we shall have if it is made to revolve in a plane to which the other axis is perpendicular; as, for instance, let x revolve in a plane at right angles to the axis of y, then, whenever x is affected with $\pm(-)^{\frac{1}{2}}$, it is to be measured in a plane passing through the axis of y, and perpendicular to the axis of x. Similarly if y be affected with $\pm(-)^{\frac{1}{2}}$, it is to be drawn in the plane passing through the axis of x, and perpendicular to the axis of y. Thus it appears that an equation between x and y not only represents a curve in the plane of the paper, but also curves in the planes at right angles to it, passing through the axes of x and y.

Let us consider the following examples:

The equation to the ellipse, referred to its centre as origin, and principal axes as coordinate axes, is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;$$

whence we have $$y = \pm\frac{b}{a}\{a^2-x^2\}^{\frac{1}{2}}, \tag{3}$$

and $$x = \pm\frac{a}{b}\{b^2-y^2\}^{\frac{1}{2}}; \tag{4}$$

and therefore neither y nor x is affected with $\pm\sqrt{-}$ as long as x is less than $\pm a$, and y is less than $\pm b$.

But let x be greater than $\pm a$, then we may write (3) in the form

$$y = \pm\sqrt{-}\frac{b}{a}\{x^2-a^2\}^{\frac{1}{2}}; \tag{5}$$

which equation, short of the symbol $\sqrt{-}$, represents an hyperbola whose transverse axis is $2a$, and conjugate axis $2b$, and whose asymptotes are as drawn in fig. 30; but which hyperbola, when the $\sqrt{-1}$ is introduced, is in the plane containing the

line A′OA, and perpendicular to the plane of the paper, and which is delineated by the dotted line; also as the equations to the asymptotes are

$$y = \pm \sqrt{-1}\,\frac{b}{a}\,x, \tag{6}$$

they lie in the same plane as the curves, and are represented by the lines OL and OL′.

Similarly, when y is greater than $\pm b$, we have

$$x = \pm \sqrt{-1}\,\frac{a}{b}\,(y^2-b^2)^{\frac{1}{2}}, \tag{7}$$

which represents an hyperbola in the plane passing through the line BOB′ and perpendicular to the plane of the paper, and which is delineated by the dotted lines of the figure, viz. SBS′ and TB′T′; and the equations to its asymptotes are

$$x = \pm \sqrt{-}\,\frac{a}{b}\,y. \tag{8}$$

Thus the general equation to the ellipse, viz. (3), not only represents the ellipse in the plane of the paper, but also two hyperbolæ in planes containing the coordinate axes and perpendicular to the plane of reference.

Similarly the equation

$$x^2 + y^2 = a^2,$$

in addition to the circle in the plane of xy, expresses also two rectangular hyperbolæ in planes perpendicular to it, and containing the axes of x and y.

Again, consider the equation to the parabola, viz.:

$$y^2 = 4mx;$$

$$\therefore \; y = \pm 2(mx)^{\frac{1}{2}},$$

and therefore the curve is in the plane of the paper for all *positive* values of x; but let x be negative, and we have

$$y = \pm \sqrt{-}\,(mx)^{\frac{1}{2}},$$

which expresses another equal parabola, but turned in the opposite direction and in a plane perpendicular to that of the paper, as is indicated by the dotted curve of fig. 31.

Many other examples of the same kind will occur in the sequel. The explanation of such "impossible" symbols is

necessary to a due adjustment of geometrical interpretation to the law of continuity; for no *algebraical* formula can (as far as is known) give points of discontinuity, neither therefore ought the geometrical representative to exhibit such; but it does so, unless we interpret those quantities which are affected with $\pm\sqrt{-1}$.

The preceding Section is but a mere sketch of a method of extensive application, and of only one part of it, viz. of that which relates to $\pm\sqrt{-}$; but in solving cubic equations, and tracing the curves which they represent, we shall meet with such symbols as $(+)^{\frac{1}{3}}$, $(-)^{\frac{1}{3}}$, &c., which indicate that the branches of the curve exist in planes inclined at 120°, &c. to the plane of the paper; but the full development would occupy more space than can be given to it in an Elementary Treatise.

Section 3.—*On the Generation of some Plane Curves of higher orders, and on their Equations.*

166.] The reader is supposed to be familiar with the principal properties of the straight line, the circle, and the three conic sections as exhibited in their algebraical equations; yet, as more copious illustration will be required in the succeeding Chapters than they afford, it is necessary to insert an account of the modes of description and the equations of some curves of a higher order; most of which too possess no small historical interest from the labour bestowed on them by ancient mathematicians.

In the first place let it be observed, that the equation to a curve may sometimes be concisely and elegantly expressed by means of a subsidiary angle: the elimination of which from the given equations will produce the better-known equation to the curve.

Thus the equation to the ellipse may be put in the forms:

$$\left.\begin{aligned} x &= a\cos\theta \\ y &= b\sin\theta \end{aligned}\right\} \quad \therefore \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \tag{9}$$

The hyperbola may be expressed by

$$\left.\begin{aligned} x &= a\sec\theta \\ y &= b\tan\theta \end{aligned}\right\} \quad \therefore \quad \frac{x^2}{a^2}-\frac{y^2}{b^2}=1. \tag{10}$$

Other examples of the same kind will be found in the sequel.

167.] The Cissoid of Diocles.

DEF.—If at equal distances from O and A, the two extremities of a diameter of a circle, two ordinates MQ and NS are drawn, and if OS be drawn cutting MQ in P, the locus of the point P is the Cissoid of Diocles.

Fig. 34. Let OC = CB = CA = the radius of the circle; OM = x, MP = y.

Then, by the geometry, OM : MP :: ON : NS;

but $\quad$ ON = OA − AN = OA − OM = $2a-x$,

NS = MQ = $\{$OM × MA$\}^{\frac{1}{2}} = (2ax-x^2)^{\frac{1}{2}}$;

substituting which values in the above proportion, we have

$$\frac{y}{x} = \frac{\{2ax-x^2\}^{\frac{1}{2}}}{2a-x};$$

$$y^2 = \frac{x^3}{2a-x}, \tag{11}$$

$$y = \pm\frac{x^{\frac{3}{2}}}{\{2a-x\}^{\frac{1}{2}}}. \tag{12}$$

The equation represents the curve described in fig. 34: the dotted part being that out of the plane of the paper, and when y is affected with $\pm\sqrt{-}$; and as the equation to the fundamental curve, viz. the circle, also expresses a rectangular hyperbola out of the plane of reference, that part arises from the hyperbola having been operated upon in a manner analogous to the circle in the above generation of the curve.

The equation will be subsequently completely analysed (see Ex. 6, Art. 211); but certain salient points of it are at once evident from the geometrical description. Thus the curve lies equally above and below the axis of x; it passes through O and B, and has for an asymptote the line drawn through A and perpendicular to OCA; it lies out of the plane of the paper to the left of O and to the right of A; in the fig. OE = a.

The polar equation to the curve is

$$r = 2a\sin\theta\tan\theta. \tag{13}$$

168.] The Witch of Agnesi.

Def.—The ordinate of a circle MQ is produced to P, so that MP : MQ :: OA : OM, the locus of the point P is the Witch of Agnesi.

Fig. 35. Let OC = CA = a, OM = x, MP = y.

Then, by the definition, MP : MQ :: OA : OM;

$$\text{but} \qquad \text{MQ} = \{2ax-x^2\}^{\frac{1}{2}};$$

$$\therefore \quad y : \{2ax-x^2\}^{\frac{1}{2}} :: 2a : x;$$

$$\therefore \quad y^2 = 4a^2\frac{2a-x}{x}, \tag{14}$$

$$y = \pm 2a\left\{\frac{2a-x}{x}\right\}^{\frac{1}{2}}. \tag{15}$$

The equation expresses the curve delineated in fig. 35, of which the dotted parts are out of the plane of reference, and arise from an analogous operation being performed on that rectangular hyperbola, out of the plane of the paper, which the equation to the fundamental curve also represents.

Although we are obliged to reserve the complete discussion of equation (15) until the next Chapter, (see Ex. 7, Art. 211,) yet it appears that the curve cuts into the axis of x at A, and that the axis of y is an asymptote; and that the ordinate is affected with $\pm\sqrt{-}$ whenever x is negative, and whenever x is greater than $2a$; in the fig. OB = OB$'$ = $2a$.

169.] The Conchoid of Nicomedes.

Def.—A point A and a straight line EOE$'$ being given, from A a straight line AQP is drawn cutting OE$'$ in Q, and P is such that QP is always equal to a given straight line OB; the locus of the point P, in the different positions of AQP, is the Conchoid of Nicomedes.

From A draw (fig. 36) AO at right angles to EOE$'$, and let OA = a; let the straight line QP = OB = b, OM = x, MP = y.

$$\text{Then, by the geometry, } \frac{\text{AO}}{\text{OQ}} = \frac{\text{MP}}{\text{MQ}} = \frac{\text{AO}+\text{MP}}{\text{OQ}+\text{MQ}} = \frac{a+y}{x};$$

$$\therefore \quad \text{OQ} = \frac{ax}{a+y}.$$

Also $$\text{PQ}^2 = \text{MP}^2 + \text{MO}^2,$$

$$b^2 = y^2 + \left(x - \frac{ax}{a+y}\right)^2;$$

$$\therefore \quad x^2 y^2 = (b^2 - y^2)(a+y)^2, \tag{16}$$

$$x = \pm \frac{a+y}{y}(b^2 - y^2)^{\frac{1}{2}}. \tag{17}$$

From which equation it appears that $x = \infty$ when $y = 0$, and therefore that EOE′ is an asymptote.

The line EOE′ is called the *rule* of the conchoid, and PQ or OB the *modulus*. If the line b is measured from O towards A instead of along AQ produced, then another curve is generated which is called the Inferior Conchoid, and is represented in the figure:

(1) If b be less than a, the upper and lower conchoids, as shewn in the figure, are somewhat similar in form.

(2) If $b = a$, the lower conchoid passes through A, and is somewhat like the lower conchoid drawn in the figure, but without the loop.

(3) If b be greater than a, the lower conchoid has an oval or loop, a double point of which is at A, and is that drawn in the figure.

The equation may be easily expressed in polar coordinates.

Let A be the pole and PAO $= \theta$, AP $= r$;

$$\therefore \quad r = a \sec\theta \pm b, \tag{18}$$

the upper and lower signs referring respectively to the upper and lower conchoids.

170.] The Lemniscata of James Bernoulli.

DEF.—From the centre of an equilateral hyperbola perpendiculars are drawn to the tangents, the locus of the point of intersection is the Lemniscata.

Fig. 37. Let x and y be the current coordinates to the lines PQ and OP, and x' y' be the coordinates to Q, the point on the hyperbola at which the tangent is drawn; the equations to the hyperbola and the tangent are

$$x'^2 - y'^2 = a^2, \tag{19}$$

$$xx' - yy' = a^2, \tag{20}$$

whence the equation to OP is $y = -\frac{y'}{x'}x$;

$$\therefore \quad \frac{x}{x'} = -\frac{y}{y'}; \tag{21}$$

and multiplying each term of (20) by one or other of these equalities, we have

$$x^2 + y^2 = \frac{a^2 x}{x'} = -\frac{a^2 y}{y'};$$

$$\therefore \quad \left.\begin{aligned} x' &= \frac{a^2 x}{x^2 + y^2} \\ y' &= -\frac{a^2 y}{x^2 + y^2} \end{aligned}\right\} \quad \therefore \text{ by means of (19);}$$

$$\therefore \quad (x^2 + y^2)^2 = a^2 (x^2 - y^2). \tag{22}$$

The curve, as is manifest from the generation of it, consists of two ovals, meeting in a double point at O; the tangents to which are coincident with the asymptotes of the hyperbola, and form angles of 45° on each side of OA.

The polar equation is

$$r^2 = a^2 \cos 2\theta. \tag{23}$$

171.] The Logarithmic Curve, fig. 32.

No better definition of the curve can be given than that expressed by its equation,

$$y = a^x. \tag{24}$$

Hence, when $x = 0$, $y = a^0 = 1$; when $x = 1$, $y = a$; when $x = \infty$, $y = \infty$; when $x = -\infty$, $y = 0$.

Therefore OA $= 1$; and as the ordinate recedes further from Oy, it increases and ultimately becomes infinite; and as x decreases (that is, increases negatively) y decreases, and the axis of x is an asymptote to the curve.

172.] The Catenary, fig. 33.

The catenary is the curve in which a perfectly flexible and uniform, though heavy, string hangs, when suspended in vacuo from two points.

Let OM $= x$, MP $= y$, OC $= c$; its equation is

$$y = \frac{c}{2}\left\{e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right\}, \tag{25}$$

where e is the Napierian logarithmic base; but as a knowledge of mechanics is requisite for a determination of the equation, the discussion of it must be reserved to a future part of our work. It is manifest however that when $x = 0$, $y = c$; and that as the equation is unaltered when $-x$ is written for $+x$, the curve is symmetrical with respect to the axis of y.

173.] The Tractory, or Equitangential Curve.

DEF.—If AP (fig. 38) be a curve, such that PT, the length of the tangent intercepted between the point of contact and the axis of x, is always equal to OA, then the locus of P is the equitangential curve.

Let OM $= x$, MP $= y$, OA $=$ PT $= a$; then the definition of the curve above given leads, as will be seen in the next Chapter, to an equation of the form

$$\frac{dy}{dx} = -\frac{y}{\{a^2-y^2\}^{\frac{1}{2}}}; \tag{26}$$

and the equation to the curve is that of which (26) is the derived-function, and is therefore

$$x = a\log\left\{\frac{a+(a^2-y^2)^{\frac{1}{2}}}{y}\right\} - \{a^2-y^2\}^{\frac{1}{2}}. \tag{27}$$

This curve is sometimes considered as generated by attaching one end of a string of constant length a to a weight at A, and by moving the other end along Ox; the weight is supposed to trace out the curve: hence arises the name *Tractory* or *Tractrix*. But the mode of generation is incorrect, unless we also consider the friction produced by traction to be infinitely great, so that the weight's momentum, which is caused by its motion, may be instantly destroyed.

174.] The Cycloid; see figs. 39 and 40.

DEF.—A cycloid is the curve traced out by a point in the circumference of a circle, as the circle rolls along a fixed straight line.

(a) Let the given straight line (fig. 39) be taken as the axis of x, and the radius of the rolling circle be a, and the origin be at the point O, when the generating point P was in contact with the fixed line; and let RPQ be a position of the generating

circle, such that OQ is equal to the arc PQ. Let E be the point in the line OAE, at which the generating point is again in contact with it, so that OAE = the circumference of the circle $= 2\pi a$. Bisect OE in A, and at A draw the ordinate AB $= 2$CP; then by the geometry of the figure, B is the highest point.

Let OM $= x$, MP $= y$, CP $=$ CQ $= a$, PCQ $= \theta$.

Then, since OQ = the arc PQ, OQ $= a\theta$;

$$\begin{aligned} \because\quad x &= \text{OM}, & y &= \text{MP}, \\ &= \text{OQ} - \text{MQ}, & &= \text{CQ} - \text{QN}, \\ &= a\theta - a\sin\theta, & &= a - a\cos\theta; \end{aligned}$$

$$\therefore \left.\begin{aligned} x &= a(\theta - \sin\theta) \\ y &= a \operatorname{versin}\theta \end{aligned}\right\}; \tag{28}$$

which two equations are those to the cycloid, when the starting-point is the origin; and if θ be eliminated, we have

$$x = a \operatorname{versin}^{-1}\frac{y}{a} - \{2ay - y^2\}^{\frac{1}{2}}. \tag{29}$$

Since $\sin\theta$ and $\operatorname{versin}\theta$ have the same values whenever θ is increased by 2π, or by 4π, ... it appears from (28) that the values of y recur whenever x is increased by $2\pi a$, or by $4\pi a$...; hence there is a series of curves similar and equal to OBE placed along the straight line OAE, parts of which at O and E are drawn in the figure; this also follows plainly from the mode of generation of the curve.

The line OAE is called *the base*, and AB *the axis*, and B *the highest point* of the cycloid.

(β) It is also frequently convenient to refer the cycloid to the highest point as origin, and to *its axis* as the axis of x, in which case its equation may be found as follows:

Fig. 40. Let RPT′ be the circle in its generating position, P being the generating point, the arc PR being equal to the line BR; from P let MP be drawn at right angles to OA, and let OQA be a semicircle described on the axis OA.

Let OM $= x$, MP $= y$, OC $=$ CQ $=$ CA $= a$, QCO $= \theta$.

Then since AB = semi-circumference = RPT′, of which the parts RB and arc RP are equal, therefore AR = the arc PT′ =

the arc OQ, on account of the similar positions and equality of the two semicircles; whence

$$\begin{aligned} y &= \text{MP}, & x &= \text{OM}, \\ &= \text{MN} + \text{NP}, & &= \text{OC} - \text{CM}, \\ &= \text{AR} + \text{MQ}, & &= a - a\cos\theta, \\ &= \text{arc OQ} + \text{CQ} \sin \text{QCM}, & &= a \operatorname{versin}\theta, \\ &= a\theta + a\sin\theta. \end{aligned}$$

Whence we have,

$$\left.\begin{aligned} y &= a(\theta + \sin\theta) \\ x &= a(1 - \cos\theta) \end{aligned}\right\}. \tag{30}$$

Which two equations, taken simultaneously, are those to the cycloid, and by the elimination of θ we have

$$y = a \operatorname{versin}^{-1}\frac{x}{a} + \{2ax - x^2\}^{\frac{1}{2}}. \tag{31}$$

175.] The Companion to the Cycloid.

It appears from the first of equations (30) of the last Article, that the ordinate to the cycloid is equal to the sum of the ordinate of the circle, (viz. MQ of fig. 40,) and a part produced (viz. PQ) which is equal to the intercepted arc OQ; but if the ordinate to a circle be produced until *the whole* is equal to the intercepted arc of the circle, the locus of the extremity is called the Companion to the Cycloid; fig. 41.

Let $\text{OM} = x$, $\text{MP} = y$, $\text{OC} = \text{CA} = a$, $\text{QCO} = \theta$; then, since MP is equal to the arc OQ,

$$\left.\begin{aligned} y &= a\theta \\ x &= a(1 - \cos\theta) \end{aligned}\right\}; \tag{32}$$

which two equations are those to the companion to the cycloid; and, if θ be eliminated, we have

$$y = a \operatorname{versin}^{-1}\frac{x}{a}. \tag{33}$$

176.] Epitrochoidal and Hypotrochoidal Curves.

DEF.—An epitrochoid is the curve generated by a point within or without the circumference of a circle, which rolls on another circle of given radius.

If the generating circle rolls *inside* the given circle, the generated curve is called the Hypotrochoid.

And if the generating point be on the circumference of the rolling circle, it is called an Epicycloid or Hypocycloid, according as it rolls without or within the fixed circle.

We shall consider the epitrochoid to be the *normal* case, and deduce the equations to the other curves from its equations by changing the signs and values of the constants.

Let O, the centre of the fixed circle, be the origin, and Q be the centre of the generating circle, and P the generating point; and suppose B, in the line QBP, to have been originally in contact with the fixed circle at A, and let OA be the axis of x; see fig. 42.

Let $\text{OM} = x$, $\text{MP} = y$, $\text{QOA} = \theta$, $\text{OR} = \text{OA} = a$, $\text{QR} = \text{QB} = b$, $\text{QP} = mb$.

Then, since one circle rolls on the other, the arc AR = the arc BR;

$$\therefore \quad \text{RQB} = \frac{a}{b}\,\theta,$$

$$\text{QPL} = 180^\circ - \frac{a+b}{b}\,\theta;$$

$$\begin{aligned} \therefore \quad x &= \text{OM}, \\ &= \text{ON} + \text{LP}, \\ &= (a+b)\cos\theta - mb\cos\left(\frac{a+b}{b}\right)\theta; \qquad (34) \\ y &= \text{MP}, \\ &= \text{NQ} - \text{QL}, \\ &= (a+b)\sin\theta - mb\sin\left(\frac{a+b}{b}\right)\theta; \qquad (35) \end{aligned}$$

and (34) and (35), taken simultaneously, are the equations to the epitrochoid.

If the generating circle rolls inside instead of outside the fixed circle the sign of b must be changed, and the curve is an hypotrochoid, the equations to which are, fig. 43,

$$\left.\begin{aligned} x &= (a-b)\cos\theta + mb\cos\frac{a-b}{b}\,\theta \\ y &= (a-b)\sin\theta - mb\sin\frac{a-b}{b}\,\theta \end{aligned}\right\}. \qquad (36)$$

177.] And if $m = 1$, the generating point is on the circumference of the rolling circle, and the curves become respectively the epicycloid and hypocycloid; and the equations are

$$\left.\begin{aligned} x &= (a+b)\cos\theta - b\cos\left(\frac{a+b}{b}\right)\theta \\ y &= (a+b)\sin\theta - b\sin\left(\frac{a+b}{b}\right)\theta \end{aligned}\right\}, \tag{37}$$

and

$$\left.\begin{aligned} x &= (a-b)\cos\theta + b\cos\left(\frac{a-b}{b}\right)\theta \\ y &= (a-b)\sin\theta - b\sin\left(\frac{a-b}{b}\right)\theta \end{aligned}\right\}; \tag{38}$$

the curves expressed by which are those dotted respectively in figs. 42 and 43.

When a and b are commensurable numbers, the branches of the curve re-enter after a certain number of revolutions of the generating circle: in which cases the subsidiary angle θ may be eliminated, and the equation expressed in an algebraical form; but when a and b are incommensurable, the branches never re-enter, and the equation can only be expressed in a transcendental form equivalent to the above equations.

Some varieties of the above curves, in which the equations assume particular forms, are subjoined.

178.] Suppose that the generating circle of the epicycloid is equal to the fixed circle, then $a = b$, and equations (37) become

$$x = 2a\cos\theta - a\cos 2\theta,$$

$$y = 2a\sin\theta - a\sin 2\theta;$$

whence, squaring and adding,

$$x^2 + y^2 = 5a^2 - 4a^2\cos\theta,$$

$$x^2 + y^2 - a^2 = 4a^2(1-\cos\theta).$$

Again,
$$x = 2a\cos\theta - a\{2(\cos\theta)^2 - 1\},$$

$$\therefore\ x - a = 2a\cos\theta(1-\cos\theta);$$

$$y = 2a\sin\theta(1-\cos\theta),$$

$$\therefore\ (x-a)^2 + y^2 = 4a^2(1-\cos\theta)^2,$$

$$\therefore\ (x^2+y^2-a^2)^2 = 4a^2\{(x-a)^2+y^2\}, \tag{39}$$

which is the equation to the curve expressed in rectangular

coordinates, o, the centre of the fixed circle, being the origin. Fig. 44.

Let us change the origin to A, and transform the equation to polar coordinates by putting $x = a + r\cos\phi$, $y = r\sin\phi$; whence

$$r = 2a(1-\cos\phi); \tag{40}$$

the curve is called the Cardioid, from its heart-like shape.

179.] In the equations to the hypocycloid, let $b = \frac{a}{4}$; in which case equations (38) become

$$x = \frac{a}{4}\left\{3\cos\theta + \cos 3\theta\right\} = a(\cos\theta)^3,$$

$$y = \frac{a}{4}\left\{3\sin\theta - \sin 3\theta\right\} = a(\sin\theta)^3;$$

$$\therefore \quad x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}; \tag{41}$$

see fig. 45.

180.] In the equations to the hypotrochoid, let $b = \frac{a}{2}$; in which case equations (36) become

$$\left.\begin{aligned} x &= \frac{a}{2}(1+m)\cos\theta \\ y &= \frac{a}{2}(1-m)\sin\theta \end{aligned}\right\}, \tag{42}$$

$$\frac{x^2}{\frac{a^2}{4}(1+m)^2} + \frac{y^2}{\frac{a^2}{4}(1-m)^2} = 1; \tag{43}$$

which equation represents an ellipse, the axes of which are

$$a(1+m) \quad \text{and} \quad a(1-m);$$

see fig. 46.

181.] In equations (38) of the hypocycloid, let $b = \frac{a}{2}$, whence we have

$$\left.\begin{aligned} x &= a\cos\theta \\ y &= 0 \end{aligned}\right\}; \tag{44}$$

which equations express a straight line on the axis of x, of length $2a$, which is coincident with the diameter of the circle.

CHAPTER X.

ON PROPERTIES OF PLANE CURVES, AS DEFINED BY EQUATIONS REFERRED TO RECTANGULAR COORDINATES.

SECTION 1.—*On Tangents and Normals, and their Properties.*

182.] To find the Equation to the Tangent to a given Curve.

DEF.—The tangent to a curve is that straight line which passes through two points of a curve infinitesimally near to each other.

In the following discussion we shall find it convenient to use sometimes the explicit, and sometimes the implicit, form of the equation to a curve; the general forms being

$$y = f(x), \tag{1}$$

$$u = \mathrm{F}(x, y) = c. \tag{2}$$

Firstly, consider the equation in the explicit form.

Let η and ξ be the current coordinates to the tangent; and suppose the straight line at first to pass through two points (x, y) $(x + \Delta x,\ y + \Delta y)$ at a finite distance apart: then the equation to such a line is

$$\eta - y = \frac{y + \Delta y - y}{x + \Delta x - x}(\xi - x), \tag{3}$$

or

$$\eta - y = \frac{\Delta y}{\Delta x}(\xi - x). \tag{4}$$

Suppose the two points to approach infinitesimally near to each other, in which case $\frac{\Delta y}{\Delta x}$ becomes $\frac{dy}{dx}$, and the line whose equation is (4) becomes a tangent; and we have

$$\eta - y = \frac{dy}{dx}(\xi - x), \tag{5}$$

or, as it may be written,

$$\frac{\eta-y}{dy} = \frac{\xi-x}{dx}. \tag{6}$$

If therefore the equation to a curve is

$$y = f(x),$$

$$\frac{dy}{dx} = f'(x),$$

and the equation to the tangent is

$$\eta - y = f'(x)\,(\xi - x). \tag{7}$$

Thus, for instance, the equation to the parabola is

$$y = 2m^{\frac{1}{2}}\,x^{\frac{1}{2}},$$

$$\therefore \quad \frac{dy}{dx} = \frac{m^{\frac{1}{2}}}{x^{\frac{1}{2}}};$$

and the equation to the tangent is

$$\eta - y = \left(\frac{m}{x}\right)^{\frac{1}{2}}(\xi - x). \tag{8}$$

183.] If the equation to the curve be given in the implicit form

$$u = \mathrm{F}(x, y) = c,$$

$$\mathrm{D}u = \left(\frac{d\mathrm{F}}{dx}\right) dx + \left(\frac{d\mathrm{F}}{dy}\right) dy = 0; \tag{9}$$

$$\therefore \quad \frac{dy}{dx} = -\frac{\left(\frac{d\mathrm{F}}{dx}\right)}{\left(\frac{d\mathrm{F}}{dy}\right)}; \tag{10}$$

of which if a substitution be made in (5), the equation to the tangent becomes

$$(\xi - x)\left(\frac{d\mathrm{F}}{dx}\right) + (\eta - y)\left(\frac{d\mathrm{F}}{dy}\right) = 0; \tag{11}$$

and if the equation to the curve be an homogeneous function of n dimensions, then by the property of such functions, proved in Art. 76, equation (112), we have

$$x\left(\frac{d\mathrm{F}}{dx}\right) + y\left(\frac{d\mathrm{F}}{dy}\right) = n\mathrm{F}, \tag{12}$$

and the equation to the tangent becomes

$$\xi\left(\frac{d\text{F}}{dx}\right) + \eta\left(\frac{d\text{F}}{dy}\right) = n\text{F}. \tag{13}$$

Generally, either (11) or (13) is the most convenient form for the equation of the tangent.

Thus the equation to the ellipse being given in the form

$$\text{F}(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

$$\left(\frac{d\text{F}}{dx}\right) = \frac{2x}{a^2}, \quad \left(\frac{d\text{F}}{dy}\right) = \frac{2y}{b^2},$$

and the equation is homogeneous and of two dimensions; $\therefore\ n = 2$, and (13) becomes, after division by 2,

$$\frac{x\xi}{a^2} + \frac{y\eta}{b^2} = 1.$$

184.] To find the Equation to the Normal to a Plane Curve at a given point.

Def.—The normal to a plane curve at a given point is the straight line perpendicular to the tangent, and which passes through the point of contact.

Let ξ, η be the current coordinates to the normal, and x, y the coordinates to the point of contact; then the equation to a line passing through (x, y) and perpendicular to that whose equation is (5), is

$$\eta - y = -\frac{dx}{dy}(\xi - x); \tag{14}$$

which may also be written in the form

$$(\eta - y)\, dy + (\xi - x)\, dx = 0. \tag{15}$$

And if the equation to the curve be an implicit function, it becomes

$$\frac{\eta - y}{\left(\frac{d\text{F}}{dy}\right)} = \frac{\xi - x}{\left(\frac{d\text{F}}{dx}\right)}. \tag{16}$$

Thus, if the equation to an ellipse be

$$\text{F}(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

the equation to the normal is

$$\frac{b^2}{y}(\eta - y) = \frac{a^2}{x}(\xi - x).$$

If the equation to the curve be algebraical and of n dimensions, the equation to the normal, as is plain from the form of (16), is also of n dimensions; to a given curve therefore of n dimensions there *may* be drawn n^2 normals through a given point.

185.] Hence also it follows, that the equation to a line passing through the origin, and perpendicular to the tangent, is

$$\frac{\eta}{\left(\frac{dF}{dy}\right)} = \frac{\xi}{\left(\frac{dF}{dx}\right)}; \tag{17}$$

by means of which, in combination with equation (11), and that to the curve, the locus of the point of intersection of the tangent, with the perpendicular on it from the origin, may be determined.

186.] Let Δs be the distance between the two points on the curve through which the cutting line of Art. 182 passes, that is, let it be the length of the chord joining them; then

$$\Delta s^2 = \Delta x^2 + \Delta y^2, \tag{18}$$

and let the two points approach infinitesimally near to one another; in which case, according to the notation of Art. 17, Δx, Δy, Δs become respectively dx, dy, ds, and we have

$$ds^2 = dx^2 + dy^2, \tag{19}$$

and ds becomes the distance between these two points, which are infinitesimally near to each other; that is, it becomes an element of the curve, or an infinitesimal arc: it is in fact the small portion of the tangent line which is common to the tangent and the curve. Or, under another mode of considering the curve, that is, of conceiving it to be generated by a point moving according to a given law, ds is the distance between two successive positions of the point; and if these two positions are taken so near to each other, that only an infinitesimal instant of time has elapsed during the passage from one to the other, it is impossible to conceive but that the moving

point has passed in a straight line from one to the other; the length of which straight line is ds.

If then we use the character τ to symbolize the angle made by the tangent with the axis of x, that is, the angle PTM in fig. 47, we have, from equation (5),

$$\tan \tau = \frac{dy}{dx}; \tag{20}$$

whence

$$\frac{dy}{\sin \tau} = \frac{dx}{\cos \tau} = \pm ds, \tag{21}$$

the last equality following from Preliminary Theorem I; the numerator and denominator of the preceding equalities having been squared and added, and subsequently the square root having been extracted.

Also again, by equation (10),

$$\frac{dy}{\pm\left(\frac{dF}{dx}\right)} = \frac{dx}{\mp\left(\frac{dF}{dy}\right)} = \frac{ds}{\left\{\left(\frac{dF}{dx}\right)^2 + \left(\frac{dF}{dy}\right)^2\right\}^{\frac{1}{2}}}; \tag{22}$$

therefore, from (21) and (22),

$$\cos \tau = \pm \frac{dx}{ds} = \frac{\pm\left(\frac{dF}{dy}\right)}{\left\{\left(\frac{dF}{dx}\right)^2 + \left(\frac{dF}{dy}\right)^2\right\}^{\frac{1}{2}}}; \tag{23}$$

$$\sin \tau = \pm \frac{dy}{ds} = \frac{\mp\left(\frac{dF}{dx}\right)}{\left\{\left(\frac{dF}{dx}\right)^2 + \left(\frac{dF}{dy}\right)^2\right\}^{\frac{1}{2}}}. \tag{24}$$

Hence also, if ψ be the angle between the normal and the axis of x, viz. the angle PGM, in fig. 47,

$$\tan \psi = \frac{dx}{dy}; \tag{25}$$

$$\therefore \quad \sin \psi = \pm \frac{dx}{ds} = \frac{\pm\left(\frac{dF}{dy}\right)}{\left\{\left(\frac{dF}{dx}\right)^2 + \left(\frac{dF}{dy}\right)^2\right\}^{\frac{1}{2}}}; \tag{26}$$

$$\cos \psi = \pm \frac{dy}{ds} = \frac{\mp\left(\frac{dF}{dx}\right)}{\left\{\left(\frac{dF}{dx}\right)^2 + \left(\frac{dF}{dy}\right)^2\right\}^{\frac{1}{2}}}. \tag{27}$$

e equations frequently render it convenient to ex- s the equation to a curve in terms of ds, dx, and dy: of a some examples, giving rise to differential expressions, ined.

1. $y^2 = 4mx$;

$$\therefore\ 2y\,dy = 4m\,dx,$$

$$\therefore\ \frac{dy}{2m} = \frac{dx}{y} = \frac{ds}{\{4m^2+y^2\}^{\frac{1}{2}}}, \tag{28}$$

$$\therefore\ \frac{dy}{ds} = \frac{2m}{\{4m^2+y^2\}^{\frac{1}{2}}} = \frac{m^{\frac{1}{2}}}{\{x+m\}^{\frac{1}{2}}},$$

$$\frac{dx}{ds} = \frac{y}{\{4m^2+y^2\}^{\frac{1}{2}}} = \frac{x^{\frac{1}{2}}}{\{x+m\}^{\frac{1}{2}}}.$$

x. 2. To find the relations between dx, dy and ds, in quation to the cycloid.

Let starting point be origin; therefore, by equation), Art. 174,

$$x = a\,\text{versin}^{-1}\frac{y}{a} - \{2ay-y^2\}^{\frac{1}{2}},$$

$$dx = \frac{y\,dy}{\{2ay-y^2\}^{\frac{1}{2}}}; \tag{29}$$

$$\therefore\ \frac{dx}{y} = \frac{dy}{\{2ay-y^2\}^{\frac{1}{2}}} = \frac{ds}{\{2ay\}^{\frac{1}{2}}}. \tag{30}$$

(β) Let highest point be origin; therefore, by equation (31), Art. 174,

$$y = a\,\text{versin}^{-1}\frac{x}{a} + \{2ax-x^2\}^{\frac{1}{2}}, \tag{31}$$

$$dy = \frac{(2a-x)\,dx}{\{2ax-x^2\}^{\frac{1}{2}}},$$

$$= \left\{\frac{2a-x}{x}\right\}^{\frac{1}{2}} dx; \tag{32}$$

$$\frac{dy}{\{2a-x\}^{\frac{1}{2}}} = \frac{dx}{x^{\frac{1}{2}}} = \frac{ds}{(2a)^{\frac{1}{2}}}. \tag{33}$$

Ex. 3. To find the relation between dx, dy and ds, in the equation to the catenary.

$$y = \frac{c}{2}\left\{e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right\},$$

$$dy = \frac{1}{2}\left\{e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right\} dx, \tag{34}$$

$$= \frac{\{y^2-c^2\}^{\frac{1}{2}}}{c}\, dx; \tag{35}$$

$$\therefore \quad \frac{dy}{\{y^2-c^2\}^{\frac{1}{2}}} = \frac{dx}{c} = \frac{ds}{y}. \tag{36}$$

186.] To discuss the Equations to the Tangent and the Normal.

Let us consider the curve drawn in fig. 47 to be the *typical form* of all curves; of which PT is the tangent line at the point P, PG the normal line; MT the subtangent; MG the subnormal; OY the perpendicular from origin on tangent line; OT, OT′ respectively the intercepts of the axes of x and y by the tangent line; and of which the lines, PT and PG, the parts of the tangent and normal lines intercepted between the point of contact and the axis of x, are called respectively the *tangent* and the *normal*,

$$\text{OM} = x, \qquad \text{MP} = y, \qquad \text{OY} = p,$$

$$\text{OT} = \xi_0, \qquad \text{OT}' = \eta_0.$$

Then, by the equations to the tangent, (5) and (11),

$$\text{if } \eta = 0,\; \xi_0 = x - y\,\frac{dx}{dy} = \frac{x\left(\frac{dF}{dx}\right) + y\left(\frac{dF}{dy}\right)}{\left(\frac{dF}{dx}\right)}, \tag{37}$$

$$\xi = 0,\; \eta_0 = y - x\,\frac{dy}{dx} = \frac{x\left(\frac{dF}{dx}\right) + y\left(\frac{dF}{dy}\right)}{\left(\frac{dF}{dy}\right)}; \tag{38}$$

the numerators of the last values of which admit of simplification by Euler's Theorem.

Without however deducing the values of the other geometrical lines of the figure from the above equations, we will

n in the following manner, which is preferable, as ddresses itself more directly to geometrical construction the senses.

. 47, PTM $= \tau$, PGM $= \psi$;

$$\therefore \quad \tan \text{PTM} = \tan \text{MPG} = \frac{dy}{dx}, \tag{39}$$

$$\tan \text{PGM} = \tan \text{TPM} = \frac{dx}{dy}. \tag{40}$$

erefore,

$$\left. \begin{aligned} \text{btangent} &= \text{MT} = \text{MP} \tan \text{MPT} = y \frac{dx}{dy}, \\ \text{Subnormal} &= \text{MG} = \text{MP} \tan \text{MPG} = y \frac{dy}{dx}; \end{aligned} \right\} \tag{41}$$

$$\left. \begin{aligned} \text{angent} = \text{PT} = y \sec \tan^{-1} \frac{dx}{dy} = y \left\{1 + \frac{dx^2}{dy^2}\right\}^{\frac{1}{2}} = y \frac{ds}{dy}, \\ \text{Normal} = \text{PG} = y \sec \tan^{-1} \frac{dy}{dx} = y \left\{1 + \frac{dy^2}{dx^2}\right\}^{\frac{1}{2}} = y \frac{ds}{dx}; \end{aligned} \right\} \tag{42}$$

$$\left. \begin{aligned} \xi_0 &= \text{OT} = \text{OM} - \text{MT} = x - y \frac{dx}{dy}, \\ \eta_0 &= \text{OT}' = \text{MP} - \text{PR} = y - x \frac{dy}{dx}; \end{aligned} \right\} \tag{43}$$

$$p = \text{OY} = \text{MP} \sin \text{MPT} - \text{OM} \sin \text{PTM},$$

$$= y \frac{dx}{ds} - x \frac{dy}{ds} = \frac{y\,dx - x\,dy}{\{dx^2 + dy^2\}^{\frac{1}{2}}}. \tag{44}$$

Also by means of equation (22),

$$p = \pm \frac{x \left(\frac{d\text{F}}{dx}\right) + y \left(\frac{d\text{F}}{dy}\right)}{\left\{ \left(\frac{d\text{F}}{dx}\right)^2 + \left(\frac{d\text{F}}{dy}\right)^2 \right\}^{\frac{1}{2}}}; \tag{45}$$

and therefore, if $\text{F}(x, y)$ be an homogeneous function of n dimensions,

$$p = \pm \frac{n\text{F}}{\left\{ \left(\frac{d\text{F}}{dx}\right)^2 + \left(\frac{d\text{F}}{dy}\right)^2 \right\}^{\frac{1}{2}}}.$$

Hence also we may put in another form the equation to the tangent given in equation (5),

$$\eta\, dx - \xi\, dy = y\, dx - x\, dy,$$

$$\eta \frac{dx}{ds} - \xi \frac{dy}{ds} = y \frac{dx}{ds} - x \frac{dy}{ds},$$

$$= p,$$

$$= \pm \frac{x\left(\frac{dF}{dx}\right) + y\left(\frac{dF}{dy}\right)}{\left\{\left(\frac{dF}{dx}\right)^2 + \left(\frac{dF}{dy}\right)^2\right\}^{\frac{1}{2}}}. \tag{46}$$

187.] From the equations to the tangent and normal it appears, that whenever x or y is affected with $\pm\sqrt{-}$, such signs will remain in the equations, and therefore η and ξ will be similarly affected; and therefore whenever the curve is out of the plane of reference, the tangent and normal also are.

Also, as $\frac{dy}{dx}$ is the trigonometrical tangent of the angle made with the axis of x by the tangent to the curve, if at any point (a, b), $\frac{dy}{dx}$ has a finite value, the curve at that point is inclined to the axis of x at a finite angle: and if $\frac{dy}{dx}$ is positive, then x and y are simultaneously increasing or decreasing, and the curve is such as the logarithmic curve of fig. 32, or the cycloid of fig. 40; and if $\frac{dy}{dx}$ is negative, as x increases y decreases, and *vice versâ*, and the curve is such as the equitangential curve of fig. 38.

If $\frac{dy}{dx} = 0$, the tangent, and therefore the curve at the point of contact, is parallel to the axis of x; and if $\frac{dy}{dx}$ changes its sign at such a point, there is a maximum or minimum ordinate, such as is drawn in figs. 12 and 13; but if $\frac{dy}{dx}$ does not change sign, then the form of the curve will be such as in figs. 14 and 15, according as $\frac{dy}{dx}$ is positive or negative.

If $\frac{dy}{dx} = \infty$, the tangent, and therefore the curve at the point of contact, is perpendicular to the axis of x, and may be such as is drawn in one or other of the diagrams of fig. 48; that is, if $\frac{dy}{dx}$ in passing through ∞ changes sign from $+$ to $-$, the point of the curve may be such as is represented in (α): if it changes sign from $-$ to $+$, it may be that represented in (β); but if $\frac{dy}{dx}$ does not change sign and is positive throughout, the curve is that indicated in (γ), and if it be negative throughout, it is that indicated in (δ).

If at any point of a curve $\frac{dy}{dx} = \frac{0}{0}$, that is, if $\left(\frac{d\text{F}}{dx}\right) = 0$ and $\left(\frac{d\text{F}}{dy}\right) = 0$, the direction of the tangent at the point will be indeterminate as far as the form of $\frac{dy}{dx}$ defines it; but it may be evaluated, and the means of doing so will be discussed in Section 4 of the present Chapter.

188.] The form of the equation to the tangent, given in equation (11), Art. 183, deserves closer consideration.

If the equation to the curve given in the implicit form, viz. $u = \text{F}(x, y) = c$, be of n dimensions, $\left(\frac{d\text{F}}{dx}\right)$ and $\left(\frac{d\text{F}}{dy}\right)$ cannot be of more than $n-1$ dimensions, and therefore it would appear that the equation

$$(\xi - x)\left(\frac{d\text{F}}{dx}\right) + (\eta - y)\left(\frac{d\text{F}}{dy}\right) = 0$$

may be of n dimensions.

But it cannot in fact be of more than of only $(n-1)$ dimensions. For suppose the equation to the curve to contain terms of n, of $(n-1)$, ... of 0 dimensions, the several expressions of which may be symbolized by u_n, u_{n-1}, ... u_1, u_0; so that

$$\text{F}(x, y) = u_n + u_{n-1} + \ldots\ldots + u_1 + u_0 = 0 \tag{47}$$

is the equation to the curve; then

$$\left.\begin{aligned} \left(\frac{d\text{F}}{dx}\right) &= \left(\frac{du_n}{dx}\right) + \left(\frac{du_{n-1}}{dx}\right) + \ldots\ldots + \left(\frac{du_1}{dx}\right) \\ \left(\frac{d\text{F}}{dy}\right) &= \left(\frac{du_n}{dy}\right) + \left(\frac{du_{n-1}}{dy}\right) + \ldots\ldots + \left(\frac{du_1}{dy}\right) \end{aligned}\right\}; \tag{48}$$

whence $x\left(\frac{dF}{dx}\right) + y\left(\frac{dF}{dy}\right)$ becomes

$$x\left(\frac{du_n}{dx}\right) + y\left(\frac{du_n}{dy}\right) + x\left(\frac{du_{n-1}}{dx}\right) + y\left(\frac{du_{n-1}}{dy}\right) + \dots\dots$$
$$+ x\left(\frac{du_1}{dx}\right) + y\left(\frac{du_1}{dy}\right),$$

which is by Euler's Theorem equal to

$$nu_n + (n-1)\,u_{n-1} + \dots\dots + 2u_2 + u_1; \tag{49}$$

and since from (47) we have

$$nu_n + nu_{n-1} + nu_{n-2} + \dots\dots + nu_1 + nu_0 = 0,$$

$x\left(\frac{dF}{dx}\right) + y\left(\frac{dF}{dy}\right)$ becomes equal to

$$-\{u_{n-1} + 2u_{n-2} + 3u_{n-3} + \dots\dots + (n-2)\,u_2$$
$$+ (n-1)\,u_1 + nu_0\}; \tag{50}$$

and therefore the equation to the tangent line becomes

$$\xi\left(\frac{dF}{dx}\right) + \eta\left(\frac{dF}{dy}\right) = -\{u_{n-1} + 2u_{n-2} + \dots\dots$$
$$+ (n-1)\,u_1 + nu_0\}, \tag{51}$$

and which is therefore of only $(n-1)$ dimensions.

Hence it follows, that to a curve of the nth order not more than $n(n-1)$ tangents can be drawn through a given point. For the combination of (51) with the equation to the curve will give rise to an expression of $n(n-1)$ dimensions, and which may have $n(n-1)$ roots, but cannot have more. If the $n(n-1)$ roots are all real, they refer to the points where the tangents meet the curve: but if they are impossible, such tangents cannot be drawn. Hence the proposition gives the number which the tangents drawn through a given point cannot exceed.

189.] The normal is the longest or the shortest line that can be drawn to a plane curve from a given point in its plane.

Let η_0 and ξ_0 be the coordinates to the given point, and x and y the current coordinates to the curve; then, if r be the distance between (η_0, ξ_0) and (x, y),

$$r^2 = (\xi_0 - x)^2 + (\eta_0 - y)^2;$$

$$\therefore\quad r\,dr = 0 = -(\xi_0 - x)\,dx - (\eta_0 - y)\,dy. \tag{52}$$

Let $u = \text{F}(x, y) = c$ be the equation to the curve,

$$\therefore \quad \left(\frac{d\text{F}}{dx}\right) dx + \left(\frac{d\text{F}}{dy}\right) dy = 0; \tag{53}$$

and as (52) and (53) are simultaneously true,

$$\frac{\xi_0 - x}{\left(\frac{d\text{F}}{dx}\right)} = \frac{\eta_0 - y}{\left(\frac{d\text{F}}{dy}\right)}; \tag{54}$$

and if we compare (52) with (15), or (54) with (16), it is manifest that they are respectively identical, and that therefore the longest and shortest lines coincide in direction with the normal.

190.] Illustrative examples on the preceding Section.

Ex. 1. Properties of the ellipse.

$$\text{F}(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;$$

$$\therefore \quad \left(\frac{d\text{F}}{dx}\right) = \frac{2x}{a^2}, \quad \left(\frac{d\text{F}}{dy}\right) = \frac{2y}{b^2}; \tag{54}$$

$$\therefore \quad \frac{dy}{dx} = -\frac{b^2 x}{a^2 y}.$$

Hence the equation to the tangent is

$$\frac{\xi x}{a^2} + \frac{\eta y}{b^2} = 1;$$

the equation to the normal is

$$\frac{a^2}{x}(\xi - x) = \frac{b^2}{y}(\eta - y),$$

or

$$\frac{a^2 \xi}{x} - \frac{b^2 \eta}{y} = a^2 - b^2.$$

Intercept of axis of x by tangent $= \xi_0 = \dfrac{a^2}{x}$,

. . . . y . . . . . $= \eta_0 = \dfrac{b^2}{y}$;

Subtangent $= y \dfrac{dx}{dy} = -\dfrac{a^2 y^2}{b^2 x}$,

Subnormal $= y \dfrac{dy}{dx} = -\dfrac{b^2}{a^2} x$;

Perpendicular from origin on tangent $= \dfrac{a^2 b^2}{\{b^4 x^2 + a^4 y^2\}^{\frac{1}{2}}}$.

Ex. 2. **Properties of the Cissoid of Diocles.**

The equation is
$$y^2 = \frac{x^3}{2a-x},$$
$$\therefore \quad \frac{dy}{dx} = \pm \frac{x^{\frac{1}{2}}(3a-x)}{(2a-x)^{\frac{3}{2}}};$$
therefore at the origin, $\because \frac{dy}{dx} = 0$, the curve touches the axis of x; when $x = 3a$, $\frac{dy}{dx} = 0$, and changes sign, and therefore the ordinates are severally a maximum and a minimum; when $x = 2a$, $\frac{dy}{dx} = \infty$, and therefore the curve is perpendicular to the axis of x; when $x = a$, $y = a$, $\frac{dy}{dx} = 2$, and therefore the curve cuts its fundamental circle at $\tan^{-1} 2$. These several values of the tangent are expressed geometrically in fig. 34.

Ex. 3. Let the curve be the hypocycloid whose equation is, Art. 179,
$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$$
The equation to the tangent is
$$\frac{\xi}{x^{\frac{1}{3}}} + \frac{\eta}{y^{\frac{1}{3}}} = a^{\frac{2}{3}};$$
and therefore
$$\xi_0 = a^{\frac{2}{3}}\, x^{\frac{1}{3}},$$
$$\eta_0 = a^{\frac{2}{3}}\, y^{\frac{1}{3}};$$
$$\therefore \quad \eta_0^2 + \xi_0^2 = a^{\frac{4}{3}}\,(x^{\frac{2}{3}} + y^{\frac{2}{3}}) = a^2,$$
and therefore the length of the tangent line intercepted between the coordinate axes is constant.

Ex. 4. To find the Differential Equation to the Equitangential Curve; see Art. 173.

According to the definition, the line PT of fig. 38 is to be of constant length. Let the length be a,
$$\therefore \quad a = y\left\{1 + \frac{dx^2}{dy^2}\right\}^{\frac{1}{2}},$$
$$\therefore \quad \frac{dy}{dx} = -\frac{y}{\{a^2-y^2\}^{\frac{1}{2}}};$$

the negative sign being taken because, according to the form drawn in fig. 38, y decreases as x increases.

Ex. 5. The Logarithmic Curve.

$$y = a^x,$$

$$\frac{dy}{dx} = \log_e a \; a^x,$$

$$= y \log_e a;$$

$$\therefore \quad \text{subtangent} = y\frac{dx}{dy} = \frac{1}{\log_e a} = \text{a constant}.$$

Also when $x = 0$, $y = 1$, in which case $\dfrac{dy}{dx} = \log_e a$, which is the tangent of the angle at which the curve is inclined to the axis of x, at the point where it cuts the axis of y.

Ex. 6. Properties of the Cycloid.

(a) Firstly, let the starting point be the origin; the equation to the curve is

$$x = a \operatorname{versin}^{-1}\frac{y}{a} - \{2ay - y^2\}^{\frac{1}{2}};$$

$$\therefore \quad \frac{dx}{dy} = \frac{y}{\{2ay-y^2\}^{\frac{1}{2}}},$$

$$\frac{dy}{dx} = \frac{\{2ay-y^2\}^{\frac{1}{2}}}{y} = \frac{2a-y}{\{2ay-y^2\}^{\frac{1}{2}}}.$$

Therefore in fig. 49,

$$\{2ay - y^2\}^{\frac{1}{2}} = \text{LP, and } 2a - y = \text{LQ, and } \frac{dy}{dx} = \tan \text{PTM},$$

$$\therefore \quad \tan \text{PTM} = \frac{\text{QL}}{\text{PL}} = \tan \text{QPL};$$

therefore the tangent at P also passes through the point Q.

Hence also the normal at the point P passes through G, the other extremity of the diameter of the generating circle; which is also manifest from the above value of $\dfrac{dx}{dy}$.

Length of normal $= \text{PG} = y\dfrac{ds}{dx} = \{2ay\}^{\frac{1}{2}}$, see equation (30);

$$\therefore \quad \text{PG}^2 = \text{QG} \times \text{GL}.$$

(β) Secondly, let highest point be origin; see fig. 50,

$$y = \{2ax - x^2\}^{\frac{1}{2}} + a \text{ versin}^{-1} \frac{x}{a},$$

$$\frac{dy}{dx} = \frac{2a-x}{\{2ax-x^2\}^{\frac{1}{2}}} = \frac{\{2ax-x^2\}^{\frac{1}{2}}}{x};$$

$$\therefore \quad \tan \text{PTM} = \frac{\text{MQ}}{\text{OM}} = \tan \text{QOM},$$

$\therefore$ PT is parallel to the chord OQ.

Hence also the normal PG is parallel to the chord QA.

Section 2.—*On Asymptotes to Plane Curves referred to Rectangular Coordinates.*

191.] On Rectilinear Asymptotes.

Def.—A line is said to be an asymptote to a curve, when the curve approaches continually nearer and nearer to it, but is not coincident with it within a finite distance.

From the definition it is plain that there are two classes of asymptotes, rectilinear and curvilinear, which it is convenient to discuss separately.

If the curve has asymptotes which are either the coordinate axes themselves or straight lines parallel to them, they may be determined in the following manner. If $y = \infty$, when $x = 0$, the axis of y is an asymptote; and if $y = 0$, when $x = \infty$, the axis of x is an asymptote: such are the axes of coordinates to the curve, $xy = k^2$.

Again, if $y = \infty$, when $x = a$, a line parallel to the axis of y, at a distance a from it, is an asymptote; and if $x = \infty$, when $y = b$, a line parallel to the axis of x, at a distance b from it, is an asymptote.

Thus suppose the equation to a curve to be

$$xy - ay - bx = 0;$$

then, as it may be put under either of the forms,

$$y = \frac{bx}{x-a} \quad \text{or} \quad x = \frac{ay}{y-b},$$

$$y = \infty, \text{ when } x = a; \text{ and } x = \infty, \text{ when } y = b;$$

that is, two lines parallel to the coordinate axes are asymptotes; the curve is represented in fig. 51; wherein OA $= a$, OB $= b$.

So of the logarithmic curve, (see fig. 32,) the axis of x is an asymptote; its equation is

$$y = a^x,$$

$$\therefore \quad y = 0, \text{ when } x = -\infty;$$

$\therefore$ OB is an asymptote to the branch AC.

So in the cissoid, equation (12), Art. 167, and fig. 34, $y = \infty$, when $x = 2a$, therefore the ordinate through A is an asymptote to the curve.

And in the tractory, equation (27), Art. 173, and fig. 38, $y = 0$, when $x = \infty$, and therefore the axis of x is an asymptote.

If however a curve has rectilinear asymptotes not parallel to the axes of coordinates, they are to be determined by one or other of the following methods.

192.] Method of determining asymptotes by expansion in descending powers of x.

If by any artifice, as by the Binomial Theorem, or by Maclaurin's Theorem, the equation to a curve can be expanded in a series of the form

$$y = a_1 x + a_0 + \frac{b_1}{x} + \frac{b_2}{x^2} + \dots\dots \tag{55}$$

Then, as all and every term after the first two, that is, every term which involves a negative power of x, diminishes without limit, and ultimately becomes infinitesimal, when x becomes infinity, the difference between the ordinate to the curve represented in equation (55), and that to the straight line whose equation is

$$y = a_1 x + a_0, \tag{56}$$

is infinitesimal, and the straight line represented by equation (56) is an asymptote to the curve.

And according as the first term after a_0, be it $\frac{b_1}{x}$ or $\frac{b_2}{x^2}$ ......, is positive or negative, so will the ordinate to the curve be greater or less than the ordinate to the asymptote, and the curve will be above or below the asymptote.

The equation (56) is to be constructed in the ordinary way.

And if, finally, the equation to the asymptote is affected with $\pm\sqrt{-}$, it indicates that the asymptote lies out of the plane of reference, and is asymptotic therefore to a branch of a curve similarly placed, and to be drawn according to the methods of Section 2 of the last Chapter.

193.] Ex. 1. To find the Asymptotes to the Hyperbola.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1;$$

$$y^2 = \frac{b^2}{a^2}(x^2-a^2),$$

$$\doteq \frac{b^2x^2}{a^2}\left(1-\frac{a^2}{x^2}\right),$$

$$y = \pm\frac{b}{a}x\left\{1-\frac{a^2}{2x^2}+\dots\dots\right\};$$

therefore neglecting inverse powers of x, we have

$$y = \pm\frac{b}{a}x;$$

that is, the asymptotes are two lines passing through the centre and inclined to the axis of x at $\pm\tan^{-1}\frac{b}{a}$; and as the next term in the expansion is negative, it follows that the curve lies below the asymptote.

Ex. 2. To find the Equations to the Asymptotes to the Cissoid of Diocles.

$$y^2 = \frac{x^3}{2a-x} = -x^2\left(1-\frac{2a}{x}\right)^{-1};$$

$$\therefore\quad y = \pm\sqrt{-}\,x\left(1+\frac{a}{x}+\dots\dots\right),$$

$$\therefore\quad y = \pm\sqrt{-}\,(x+a),$$

are the equations to the asymptotes, and represent two straight lines out of the plane of reference inclined to the axis of x at $\pm 45°$, and cutting the axis of x at a distance $-a$ from the origin, which are delineated by the dotted straight lines of fig. 34.

Ex. 3. **To determine the Asymptotes of the Witch of Agnesi.**

From the equation (14) of Art. 168, we have

$$y^2 = 4a^2\,\frac{2a-x}{x},$$

$$= 4a^2\left(\frac{2a}{x} - 1\right),$$

$$\therefore \quad y = \pm\sqrt{-}\,2a, \text{ when } x = \infty;$$

which equations are those to the asymptotes and express two straight lines out of the plane of reference, and parallel to the axis of x, at distances $\pm\, 2a$ from it; see fig. 35.

Ex. 4. To determine the Asymptotes of

$$y^2 = x^2\,\frac{x^2-1}{x^2+1},$$

$$y = \pm x\left(1-\frac{1}{x^2}\right)^{\frac{1}{2}}\left(1+\frac{1}{x^2}\right)^{-\frac{1}{2}},$$

$$= \pm x\left(1-\frac{1}{2x^2}-\frac{1}{8x^4}+\ldots\right)\left(1-\frac{1}{2x^2}+\frac{3}{8x^4}+\ldots\right),$$

$$= \pm x\left(1-\frac{1}{x^2}+\ldots\ldots\right);$$

$\therefore$ neglecting terms involving negative powers of x, the equations to the asymptotes are

$$y = \pm\, x;$$

and as the next term of the series is negative, it follows that the curve is (in the first quadrant) below the asymptote.

194.] Frequently however the equation to the curve is such as not without difficulty to admit of expansion in the required form, viz. equation (55); in which case the Calculus supplies the following method, and which is of universal application.

The definition of an asymptote enables us to consider it as a tangent to the curve at an infinite distance from the origin, whereby the problem resolves itself into the construction of this particular tangent; and since the tangent makes with the axis of x $\tan^{-1}\dfrac{dy}{dx}$, and its intercepts of the axes are η_0 and ξ_0,

the values of which are given in equations (37) and (38), Art. 186, the determination of two of these three quantities, corresponding to $x = \infty$ and $y = \infty$, will enable us to construct the asymptote. For the infinite values of the coordinates, these expressions will be of the form $\frac{\infty}{\infty}$, and must be evaluated according to the method of Art. 114.

On considering the values of η_0 and ξ_0 in Art. 186, it will be seen that, if $F(x, y)$ be of n dimensions, the denominators of the expressions are of $n-1$ dimensions, and that the numerators are apparently of n dimensions; whence it might be inferred, that η_0 and ξ_0 always are equal to ∞; but such is not the case; for the numerator, by means of Euler's Theorem, and by the process explained in Art. 188, is of only $n-1$ dimensions, and therefore η_0 and ξ_0 may have (as far as their algebraical values are concerned) finite values. In a particular example however care must be taken to replace, by means of the equation to the curve, the highest powers of the variables in the numerator by their equivalents of lower dimensions.

The most convenient plan is first to evaluate $\frac{dy}{dx}$, and, if it has a finite value, to determine η_0, since the equation to the asymptote is

$$\eta = \xi \frac{dy}{dx} + \eta_0.$$

And if $\frac{dy}{dx} = 0$, the asymptote is parallel to the axis of x, and at a distance from it to be determined by η_0; hence η_0 must be determined.

But if $\frac{dy}{dx} = \infty$, the asymptote is parallel to the axis of y, and ξ_0 must be evaluated in order to determine the distance from the origin at which it cuts the axis of x.

195.] To apply the above processes to examples.

Ex. 1. To determine the Equations to the Asymptotes of the Hyperbola.

$$F(x, y) = \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1;$$

$$\therefore \quad \text{when } x = \infty, \ y = \infty;$$

$$\left(\frac{dF}{dx}\right) = \frac{2x}{a^2}, \quad \left(\frac{dF}{dy}\right) = -\frac{2y}{b^2},$$

$$\frac{dy}{dx} = \frac{b^2 x}{a^2 y} = \frac{\infty}{\infty}, \text{ when } x = y = \infty;$$

luating according to the method of Art. 114,

$$\frac{dy}{dx} = \frac{b^2}{a^2 \dfrac{dy}{dx}};$$

$$\therefore \quad \frac{dy^2}{dx^2} = \frac{b^2}{a^2}, \quad \frac{dy}{dx} = \pm\frac{b}{a},$$

$$\eta_0 = \frac{x\left(\dfrac{dF}{dx}\right) + y\left(\dfrac{dF}{dy}\right)}{\left(\dfrac{dF}{dy}\right)} = -\frac{b^2}{y} = 0, \text{ when } y = \infty;$$

e the equations to the asymptotes are

$$\eta = \pm\frac{b}{a}\xi.$$

Ex. 2. To find the Asymptotes to the Curve.

$$y^3 = ax^2 - x^3;$$

$$\therefore \quad F(x, y) = y^3 - ax^2 + x^3 = 0,$$

$$\left(\frac{dF}{dx}\right) = 3x^2 - 2ax,$$

$$\left(\frac{dF}{dy}\right) = 3y^2;$$

$$\frac{dy}{dx} = \frac{2ax - 3x^2}{3y^2} = \frac{\infty}{\infty}, \text{ when } x = y = \infty,$$

$$= \frac{2a - 6x}{6y\dfrac{dy}{dx}} = \frac{\infty}{\infty}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

whence, differentiating again, and bearing in mind that $\frac{dy}{dx}$ is a definite quantity, and therefore not admitting of differentiation, we have

$$\frac{dy}{dx} = \frac{-6}{6\left(\frac{dy}{dx}\right)^2};$$

$$\therefore \quad \frac{dy}{dx} = -1, \text{ when } x = y = \infty,$$

$$\eta_0 = \frac{x\left(\frac{dF}{dx}\right) + y\left(\frac{dF}{dy}\right)}{\left(\frac{dF}{dy}\right)} = \frac{ax^2}{3y^2} = \frac{\infty}{\infty}, \text{ when } x = y = \infty,$$

$$= \frac{2ax}{6y\frac{dy}{dx}} = \frac{\infty}{\infty}, \text{ when } x = y = \infty,$$

$$= \frac{a}{3\frac{dy^2}{dx^2}} = \frac{a}{3}, \text{ when } x = y = \infty, \text{ in which case } \frac{dy}{dx} = -1;$$

$\therefore$ the equation to the asymptote is

$$\eta = -\xi + \frac{a}{3}.$$

Ex. 3. To find the Equation to the Asymptote of the Curve (the Folium of Descartes), whose equation is

$$x^3 - 3axy + y^3 = 0 = F(x, y);\ \text{see fig. 63.}$$

$$\therefore \quad \left(\frac{dF}{dx}\right) = 3x^2 - 3ay,$$

$$\left(\frac{dF}{dy}\right) = 3y^2 - 3ax,$$

$$\frac{dy}{dx} = -\frac{x^2 - ay}{y^2 - ax} = \frac{\infty}{\infty}, \text{ when } x = y = \infty,$$

$$= -\frac{2x - a\frac{dy}{dx}}{2y\frac{dy}{dx} - a} = \frac{\infty}{\infty},$$

$$= -\frac{2}{2\left(\frac{dy}{dx}\right)^2};$$

$$\therefore \quad \frac{dy}{dx} = -1, \text{ when } x = y = \infty,$$

$$\eta_0 = \frac{3axy}{3y^2-3ax} = \frac{axy}{y^2-ax} = \frac{\infty}{\infty}, \text{ when } x = y = \infty.$$

$$= \frac{ax\frac{dy}{dx} + ay}{2y\frac{dy}{dx} - a} = \frac{\infty}{\infty}, \ldots \ldots$$

$$= \frac{2a\frac{dy}{dx}}{2\left(\frac{dy}{dx}\right)^2} = -a, \text{ when } x = y = \infty, \text{ and } \frac{dy}{dx} = -1;$$

$\therefore$ the equation to the asymptote is

$$\eta = -\xi - a.$$

196.] Sometimes by this method, as well as by the former, we arrive at results affected with $\pm\sqrt{-}$, in which case the lines must be drawn in their own planes. And sometimes curves have branches out of the plane of reference, which are asymptotic to straight lines in the plane, as in the following example; and these must be determined in one or other of the methods which have been just explained.

$$x^4(y-x)^2 = a^2 - x^2,$$

$$\therefore \quad y - x = \pm\frac{(a^2-x^2)^{\frac{1}{2}}}{x^2}.$$

When x is infinitely great, the right-hand side of the equation vanishes, and we have $y = x$, which is the equation to a line passing through the origin, and inclined at 45° to the axis of x, and which is asymptotic to two branches of the curve; but, for all values of x not within the limits $\pm a$, the curve lies out of the plane of reference, and therefore the asymptote is that to which these branches are continually approaching. The form of the curve is given in fig. 52, the dotted lines representing the branches in a plane perpendicular to that of the paper.

197.] On Curvilinear Asymptotes.

Two curves may also be asymptotic to each other; for suppose that the equation to a given curve, when expanded in descending powers of x, is

$$y = a_2 x^2 + a_1 x + a_0 + \frac{b_1}{x} + \dots\dots$$

Then, if we neglect on the right-hand side of the equation all terms after the first three, which is equivalent to making $x = \infty$, the curve whose equation is

$$y = a_2 x^2 + a_1 x + a_0,$$

and which is a parabola, is asymptotic to the given curve.

Also if in equation (55), Art. 192, we take account of the first three terms only, and multiply through by x, and neglect all the subsequent terms, then we have the equation to an hyperbola, viz.

$$xy = a_1 x^2 + a_0 x + b_1;$$

and this curve is asymptotic to the given curve, because the difference between the lengths of their ordinates is a quantity which diminishes without limit as x increases without limit. And so again if we take account of the first four terms of the same equation, and neglect all subsequent ones, we shall obtain the equation to a curve which is nearer to the given curve than either the rectilinear asymptote or the hyperbola; and thus by a similar process we may obtain a series of curves more and more asymptotic to the given curve. Thus also we often find curves which have cubical and semicubical parabolas asymptotic to them: as for instance:

$$y^2 = x^2 \{x^2 - a^2\}^{\frac{1}{2}};$$

$$\therefore \quad y = \pm\, x^{\frac{3}{2}} \left\{1 - \frac{a^2}{x^2}\right\}^{\frac{1}{4}},$$

$$= \pm\, x^{\frac{3}{2}} \left\{1 - \frac{a^2}{4x^2} - \dots\dots\right\},$$

$$= \pm\, \left\{x^{\frac{3}{2}} - \frac{a^2}{4x^{\frac{1}{2}}} - \dots\dots\right\};$$

and therefore, neglecting all terms involving negative powers of x, we have

$$y = \pm\, x^{\frac{3}{2}},$$

which is the equation to a semicubical parabola (the form of which is given in fig. 64), and is asymptotic to the curve.

SECTION 3.—*On Direction of Curvature and Points of Inflexion.*

198.] The value of $\frac{dy}{dx}$, which we have discussed in the preceding Sections, enables us to determine the inclination to either of the coordinate axes of a curve at any point, but tells us nothing as to *the direction of curvature*, that is, as to whether the curve is *concave* or *convex* towards a certain line or in a given direction; we use these words in their common meaning; and we proceed to discover criteria which will determine such direction. To simplify the formulæ we shall take x to be an equicrescent variable.

Since under this supposition $\frac{d^2y}{dx^2} = \frac{d.\frac{dy}{dx}}{dx} = \frac{d.\tan\tau}{dx}$ (see equation (20), Art. 185), and since $d.\tan\tau = (\sec\tau)^2 d\tau$, which is always positive; it follows that if $\frac{d^2y}{dx^2}$ is positive, τ and x are increasing or decreasing simultaneously; and if $\frac{d^2y}{dx^2}$ is negative, as x increases, τ decreases, and *vice versâ*.

Now from the geometry it is plain, that if τ and x simultaneously increase or decrease, the form of the curve must be such as that of fig. 53, that is, must be *convex downwards;* and if as x increases τ decreases, or *vice versâ*, the figure must be such as that of fig. 54, that is, the curve must be *concave downwards*. Hence we have the following criteria of the direction of curvature.

If $\frac{d^2y}{dx^2}$ *is positive, the curve is convex downwards; and if* $\frac{d^2y}{dx^2}$ *is negative, the curve is concave downwards.*

Suppose that at a certain point (a, b), $\frac{d^2y}{dx^2}$ changes its sign, which it can do only by passing through 0 or ∞, then the direction of curvature changes; and if the change of sign be from $+$ to $-$, then the curvature, having been convex downwards, becomes concave; and if the change be from $-$ to $+$, the reverse is the case. A point where such a change of curvature takes place is called *a point of inflexion*. To determine which, we must equate $\frac{d^2y}{dx^2}$ to 0 and to ∞; and if at the cor-

responding critical value there is a change of sign, then such a point is a point of inflexion; fig. 55, (γ) and (δ) of fig. 48, fig. 14 and fig. 15 illustrate such points of inflexion.

This is also evident from the following considerations.

199.] Let $y = f(x)$ be the equation to a curve; and suppose that we consider it not only at the point (x, y), but also at another point $(x+h, y+k)$, so that

$$y + k = f(x+h); \tag{58}$$

then, expanding $f(x+h)$ to three terms by Taylor's Series (see Art. 115),

$$y + k = f(x) + f'(x)\frac{h}{1} + \frac{h^2}{1.2}f''(x+\theta h). \tag{59}$$

And if a tangent be drawn to the curve at the point (x, y), its equation is

$$\eta = y + \frac{dy}{dx}(\xi - x); \tag{60}$$

and therefore its ordinate, when ξ becomes $x+h$, is given by the equation

$$\eta = y + \frac{dy}{dx}h; \tag{61}$$

in which, replacing y by $f(x)$, and $\frac{dy}{dx}$ by $f'(x)$, and subtracting (61) from (59), we obtain the difference between the ordinates to the curve and to the tangent corresponding to the point $x+h$, and we have

$$y + k - \eta = \frac{h^2}{1.2}f''(x+\theta h); \tag{62}$$

and taking h to be infinitesimal, which is equivalent to considering only the point in the curve which is next to $(x+dx, y+dy)$, whereby $f''(x+\theta h)$ becomes $f''(x)$, that is, $\frac{d^2y}{dx^2}$, we have

$$y + k - \eta = \frac{d^2y}{dx^2}\frac{h^2}{1.2}. \tag{63}$$

And therefore if $\frac{d^2y}{dx^2}$ be positive, the ordinate to the curve is greater than the ordinate to the tangent, and therefore the curve lies above the tangent, whatever be the sign of h; that is,

convex downwards, as in fig. 53; but if $\frac{d^2y}{dx^2}$ be
tive, contrary results follow, and the curve is concave
wards, as in fig. 54.

efore at any point the curve passes through the tan-
as to be above it on one side of the point of contact
ion, but below it on the other, then $y+k-\eta$ must
sign as h changes sign, which can only be the case when
and $\frac{d^3y}{dx^3}$ is a finite quantity; or in general, when the
the expansion of $f(x+h)$, which gives sign to the
one involving an odd power of h, in which case the
s from below the tangent to above it, or *vice versâ*,
er of the manners indicated in fig. 55. It is plain
point of inflexion, $\frac{dy}{dx}$, and therefore the angle τ,
ximum or minimum value.

vature of a curve towards the right or the left may
mined in a similar manner by discussing the value
nd sign of $\frac{d^2x}{dy^2}$.

200.] **Examples illustrative of the foregoing theory.**

Ex. 1. To determine the direction of Curvature of the Curve whose Equation is

$$y = \frac{(x-1)(x-3)}{(x-2)},$$

$$\frac{dy}{dx} = \frac{x^2-4x+5}{(x-2)^2},$$

$$\frac{d^2y}{dx^2} = -\frac{2}{(x-2)^3};$$

that is, $\frac{d^2y}{dx^2}$ is positive or negative according as x is less or greater than 2; and therefore the curve is convex downwards for all values of x less than 2, and concave downwards for all values of x greater than 2; and when $x = 2$, there is a point of inflexion.

Ex. 2. To determine the direction of Curvature of the Witch of Agnesi.

$$y = 2a\left(\frac{2a-x}{x}\right)^{\frac{1}{2}},$$

$$\frac{dy}{dx} = \frac{-2a^2}{x^{\frac{3}{2}}(2a-x)^{\frac{1}{2}}},$$

$$\frac{d^2y}{dx^2} = \frac{2a^2(3a-2x)}{x(2ax-x^2)^{\frac{3}{2}}};$$

therefore the upper branch of the curve in the plane of reference is convex downwards for all values of x between 0 and $x = \frac{3a}{2}$; and when x is greater than $\frac{3a}{2}$, $\frac{d^2y}{dx^2}$ is negative, and the curve is concave downwards; at the point therefore, $x = \frac{3a}{2}$, there is a point of inflexion. This investigation explains the form of the curve as drawn in fig. 35.

Ex. 3. To prove that the Equitangential Curve (see Art. 173) is always convex downwards.

$$\frac{dy}{dx} = -\frac{y}{(a^2-y^2)^{\frac{1}{2}}};$$

$$\therefore \quad \left(\frac{dy}{dx}\right)^2 = \frac{y^2}{a^2-y^2},$$

$$\therefore \quad 2\frac{dy}{dx}\frac{d^2y}{dx^2} = \frac{2a^2y}{(a^2-y^2)^2}\frac{dy}{dx},$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{a^2y}{(a^2-y^2)^2};$$

and therefore, as y is always positive (see fig. 38), $\frac{d^2y}{dx^2}$ is always positive; and the curve is convex downwards.

Ex. 4. To determine the point of Inflexion of the Companion to the Cycloid.

$$y = a\theta, \qquad \therefore \quad dy = a\,d\theta,$$

$$x = a(1-\cos\theta), \qquad dx = a\sin\theta\,d\theta;$$

$$\frac{dy}{dx} = \operatorname{cosec}\theta,$$

$$\frac{d^2y}{dx^2} = -\operatorname{cosec}\theta\cot\theta\frac{d\theta}{dx},$$

$$= -\frac{\cos\theta}{a(\sin\theta)^3};$$

for all values of θ less than $\frac{\pi}{2}$, $\frac{d^2y}{dx^2}$ is negative, and the curve is concave downwards, and for values of θ greater than $\frac{\pi}{2}$, the curve is convex downwards; and when $\theta = \frac{\pi}{2}$, that is, when $y = \frac{\pi a}{2}$ and $x = a$, there is a point of inflexion; see fig. 41.

In a similar manner it may be shewn that the logarithmic curve and the catenary are both convex downwards.

201.] If the equation to the curve be given in the implicit form

$$F(x, y) = c,$$

then, by Art. 78, equation (122), x being equicrescent,

$$\frac{d^2y}{dx^2} = -\frac{\left(\frac{d^2F}{dx^2}\right)\left(\frac{dF}{dy}\right)^2 - 2\left(\frac{d^2F}{dx\,dy}\right)\left(\frac{dF}{dx}\right)\left(\frac{dF}{dy}\right) + \left(\frac{d^2F}{dy^2}\right)\left(\frac{dF}{dx}\right)^2}{\left(\frac{dF}{dy}\right)^3}; \quad (64)$$

and therefore for all values of the coordinates for which this quantity is positive, the curve is convex downwards: and for those for which it is negative, the curve is concave downwards; and if at any point on the curve

$$\left(\frac{d^2F}{dx^2}\right)\left(\frac{dF}{dy}\right)^2 - 2\left(\frac{d^2F}{dx\,dy}\right)\left(\frac{dF}{dx}\right)\left(\frac{dF}{dy}\right) + \left(\frac{d^2F}{dy^2}\right)\left(\frac{dF}{dx}\right)^2 = 0,$$

and changes sign, and at the same time $\left(\frac{dF}{dy}\right)$ does not $= 0$, then there is a point of inflexion.

Similarly might we find $\frac{d^2x}{dy^2}$, and determine whether a curve is convex or concave towards the right or the left.

202.] The subject next in order for discussion is the nature of those points on a curve at which $\frac{dy}{dx}$ assumes an indeterminate form; before however we proceed to it, it is advisable to examine fig. 56, which is intended to be a drawing of an infinitely-magnified continuous curve, and which is supposed to be generated by a point moving according to a continuous law; the curve is placed as in the figure in order that $\frac{d^2y}{dx^2}$ may be positive.

Let $y = f(x)$ be the equation to the curve, $\text{OK} = x$, $\text{KP} = y$, and let us consider x to be equicrescent, so that $\text{KL} = \text{LM} = \text{MN} = \dots\dots = dx$, and $d^2x = d^3x = \dots\dots = 0$; and let P, Q, R, S be points on the curve corresponding to the successive values of x; and if we conceive the successive elements of the curve, PQ, QR, RS, to be infinitely-magnified, then they are such straight lines as we have imagined ds to be in Art. 186. In the same manner then as in Art. 63,

$$\begin{aligned}
f(x) &= y = \text{KP},\\
f(x+dx) &= y+dy = \text{LQ},\\
f(x+2dx) &= y+2dy+d^2y = \text{MR},\\
f(x+3dx) &= y+3dy+3d^2y+d^3y = \text{NS},
\end{aligned}$$

and so on; whence, by subtraction, we have

$$dy = \text{LQ} - \text{KP} = \text{QU} = \text{XV} = \text{TW},$$

and $\text{MR} = \text{MZ} + \text{RZ}$: but $\text{MZ} = \text{MV} + 2\text{VX} = y + 2dy$;

$\therefore$ substituting for MR from above, we have

$$y + 2dy + d^2y = y + 2dy + \text{RZ},$$

$$\therefore \quad d^2y = \text{RZ}.$$

As we have not deduced any geometrical properties from d^3y, it is unnecessary to do more than to shew what line is represented by it.

From above we have $\text{NS} = y + 3dy + 3d^2y + d^3y$:

but $\quad \text{NW} = y, \quad \text{WG} = 3dy, \quad \text{GE} = 2d^2y,$

$$\begin{aligned}
\therefore \quad \text{SE} &= d^2y + d^3y,\\
&= \text{RZ} + d^3y;\\
\therefore \quad d^3y &= \text{SE} - \text{RZ}.
\end{aligned}$$

Hence it is manifest that $\frac{dy}{dx} = \frac{\text{QU}}{\text{PU}} = \tan \text{QPU} = \tan \text{PJO}$ $=$ trigonometrical tangent of the angle made with axis of x by the tangent to curve; and therefore if at any point on a curve $\frac{dy}{dx} = 0$, y does not increase as we pass from one point to the next consecutive point: and therefore the element of the curve PQ is along the line PU, and is parallel to the axis of x. Similarly, whenever $\frac{dy}{dx} = \infty$, the element PQ is perpendicular to PU, and parallel to the axis of y.

And since $d^2y = \text{RS}$, it is plain that d^2y represents the deflexion of the curve from the tangent line: and therefore if $d^2y = 0$, three consecutive points are in the same straight line, and the curve has for those three points become straight; and if d^2y be positive, the line RS is to be measured up from the tangent, and the curve lies above the tangent; but if d^2y be negative, it must be measured downwards, and the curve lies below the tangent; if therefore d^2y is positive on both sides of the point P, the curve is convex downwards, and is such as is drawn in fig. 53, and if d^2y is negative on both sides of the point (x, y), then the curve is concave downwards as in fig. 54; and if d^2y changes its sign at the point by passing through 0 or ∞, the curve is above the tangent on one side of the point, and below it on the other, and therefore there is a point of inflexion.

Hence we learn the geometrical meaning of the process of differentiation; it implies a passage from one point of a curve to the next consecutive point; and, as often as we differentiate, we pass to successive points, and obtain expressions which represent deflexions from straight lines and so on.

Thus, by means of one differentiation, we consider the curve with respect to two points on it; by two differentiations we consider the curve at three points, and so on. More will be said hereafter on the properties of curves under this mode of considering them; and especially in Chapters XII and XIII with respect to several successive points being common to two or more curves.

Section 4.—*On Multiple Points of Plane Curves.*

203.] Thus far we have considered the geometrical properties which belong to the first and second derived-functions of an equation to a curve when they have determinate values; but suppose that at any point on a curve, which it will be convenient as heretofore to designate by a special name, and which we will therefore call a critical point, $\frac{dy}{dx}$ assumes the indeterminate form $\frac{0}{0}$, a question occurs, and which has to be inquired into, What is the meaning of such indeterminateness? In the following discussion we shall find it most convenient to use the implicit form of the equation to the curve; viz.

$$u = \mathrm{F}(x, y) = c; \tag{65}$$

whence we have

$$\frac{dy}{dx} = -\frac{\left(\frac{d\mathrm{F}}{dx}\right)}{\left(\frac{d\mathrm{F}}{dy}\right)}. \tag{66}$$

Suppose then, at the critical point, $\frac{dy}{dx} = \frac{0}{0}$; the conditions of which are that $\left(\frac{d\mathrm{F}}{dx}\right) = 0$, $\left(\frac{d\mathrm{F}}{dy}\right) = 0$; and let the expression be evaluated according to the method of Art. 114, whereby we have

$$\frac{dy}{dx} = -\frac{\left(\frac{d^2\mathrm{F}}{dx^2}\right) dx + \left(\frac{d^2\mathrm{F}}{dx\,dy}\right) dy}{\left(\frac{d^2\mathrm{F}}{dy^2}\right) dy + \left(\frac{d^2\mathrm{F}}{dx\,dy}\right) dx},$$

$$= -\frac{\left(\frac{d^2\mathrm{F}}{dx^2}\right) + \left(\frac{d^2\mathrm{F}}{dx\,dy}\right)\frac{dy}{dx}}{\left(\frac{d^2\mathrm{F}}{dy^2}\right)\frac{dy}{dx} + \left(\frac{d^2\mathrm{F}}{dx\,dy}\right)}. \tag{67}$$

And suppose that this quantity does not become $\frac{0}{0}$ at the critical point in question, that is, that all the second partial differential coefficients do not vanish, then multiplying and reducing, we have

$$\left(\frac{d^2\mathrm{F}}{dy^2}\right)\frac{dy^2}{dx^2} + 2\left(\frac{d^2\mathrm{F}}{dx\,dy}\right)\frac{dy}{dx} + \left(\frac{d^2\mathrm{F}}{dx^2}\right) = 0, \tag{68}$$

a quadratic in $\frac{dy}{dx}$; giving therefore two values for $\frac{dy}{dx}$, and thereby shewing that two branches of the curve pass through the point, which is called a *double point*, admitting of several varieties, according as the roots of (68) are real and unequal, real and equal, or impossible; and according as the curve extends or not in the plane of reference on both sides of the point in question. Now the roots of (68) are

$$\left.\begin{array}{l}\text{real and unequal}\\ \text{real and equal}\\ \text{impossible}\end{array}\right\} \text{according as } \left(\frac{d^2 F}{dy^2}\right)\left(\frac{d^2 F}{dx^2}\right) \text{ is } \left\{\begin{array}{c}<\\=\\>\end{array}\right\} \left(\frac{d^2 F}{dx\,dy}\right)^2. \tag{69}$$

Let us first consider the case of the two roots being real and unequal; in which we have

$$\frac{dy}{dx} = \frac{-\left(\frac{d^2 F}{dx\,dy}\right) \pm \left\{\left(\frac{d^2 F}{dx\,dy}\right)^2 - \left(\frac{d^2 F}{dx^2}\right)\left(\frac{d^2 F}{dy^2}\right)\right\}^{\frac{1}{2}}}{\left(\frac{d^2 F}{dy^2}\right)}. \tag{70}$$

If the curve extends in the plane of reference on both sides of the point in question, as P_0 in fig. 57, the point is called a *real double point;* but if the curve is in the plane of reference on one side of the point, but is in another plane on the other side, as is indicated in fig. 58, where the dotted lines shew the course of the curve out of the plane of reference, such a point is called a *salient point;* and if the curve is out of the plane of reference on both sides of the point in question, but pierces the plane at the point, we have what is called a *conjugate* or *isolated point,* which however corresponds to the case of two roots of equation (68), being impossible.

Secondly, if two roots be real and equal, we have

$$\frac{dy}{dx} = -\left\{\frac{\left(\frac{d^2 F}{dx^2}\right)}{\left(\frac{d^2 F}{dy^2}\right)}\right\}^{\frac{1}{2}}, \tag{71}$$

and there are two branches passing through the point and having the same tangent.

If these branches are in the plane of reference on both sides of the point, the curve is such as one or the other of those

delineated in fig. 59, and the points where the curve meets the tangent are called points of *osculation* and by French writers *points d'embrassement;* and if they are in the plane of reference on one side of the point, and on the other side pass out of it, then the curve at the point is such as one or other of those drawn in fig. 60, where the dotted lines indicate the course of the curve out of the plane of the paper, and the points are called *cusps,* and by French writers *points de rebroussement.* That in the fig. 60, marked (α), is called a *ceratoid* cusp, or a cusp of the first species, in which the two branches touch the common tangent on opposite sides of it; that marked (β) is called a *ramphoid cusp,* or cusp of the second series, in which the two branches touch the common tangent on the same side. Cusps in other positions are shewn in fig. 61.

Thirdly, if two roots of (68) are impossible, both branches of the curve are out of the plane of reference on both sides of the point in question, but pierce it at the point; in which case there is a conjugate point, admitting of varieties according as both branches have the same or different tangent lines.

These several subordinate varieties of double points must be distinguished by examining the form and nature of the equation to the curve, and of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, when $x = a \pm h$, $y = b \pm k$, h and k being taken very small, a and b being values of the coordinates which satisfy the equation to the curve; as e.g. in fig. 60, if when $x = a + h$, $\frac{d^2y}{dx^2}$ is positive for one and negative for the other branch of the curve, and when $x = a - h$, $\frac{dy}{dx}$ is affected with $\sqrt{-}$, the curve is of the form drawn in fig. (α); but if $\frac{d^2y}{dx^2}$ is positive for both branches of the curve, and the curve is out of the plane of the paper when x is less than a, then it is such as is delineated in fig. (β).

204.] Of these several varieties of double points we shall give instances, and shall apply to particular cases the principles of the foregoing method, without adapting them directly to the general forms of the results.

To examine the nature of the point at the origin
scata whose equation is (see Art. 170 and fig. 37)

$$(x^2 + y^2)^2 = a^2(x^2 - y^2).$$

ing

$$+ y^2)(x\,dx + y\,dy) = 2a^2(x\,dx - y\,dy);$$

$$= \frac{a^2x - 2x(x^2+y^2)}{a^2y + 2y(x^2+y^2)} = \frac{0}{0}, \text{ when } x = 0, \text{ and } y = 0,$$

$$= \frac{a^2 - \ldots\ldots}{a^2\frac{dy}{dx} + \ldots} = \frac{1}{\frac{dy}{dx}}; \quad . \quad . \quad . \quad . \quad . \quad .$$

$$\therefore \quad \frac{dy^2}{dx^2} = 1, \quad \frac{dy}{dx} = \pm 1;$$

is, two branches of the curve pass through the origin, utting the axis of x at angles respectively of 45° and 135°, and which are in the plane of reference on both sides of the ; and therefore all the characteristics of a true double po are satisfied.

Ex. 2. Determine the nature of the point at the origin of the curve.

$$y^2 = x^2(1 - x^2);$$

$$\therefore \quad \frac{dy}{dx} = \frac{x - 2x^3}{y} = \frac{0}{0}, \text{ at origin,}$$

$$= \frac{1 - 6x^2}{\frac{dy}{dx}} = \frac{1}{\frac{dy}{dx}}; \quad . \quad . \quad . \quad . \quad . \quad .$$

$$\therefore \quad \frac{dy^2}{dx^2} = 1, \quad \frac{dy}{dx} = \pm 1;$$

and therefore two branches of the curve pass through the origin, which are inclined to the axis of x at angles respectively of 45° and 135°, and there is at origin a true double point; see fig. 62.

Ex. 3. To determine the nature of the point at the origin of the Folium of Descartes, the equation to which is

$$x^3 - 3axy + y^3 = 0,$$

$$\frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax} = \frac{0}{0}, \text{ at the origin,}$$

$$= \frac{a\dfrac{dy}{dx} - 2x}{2y\dfrac{dy}{dx} - a} = -\frac{dy}{dx}; \quad \ldots\ldots$$

$\therefore \ \dfrac{dy}{dx} = 0$, and $\dfrac{dy}{dx} = \infty$, and therefore there are two branches passing through the origin, and touching respectively the axes of x and y; see fig. 63.

As no form of algebraic equation can express a salient point of a curve such as we have described it, and as it can arise only from a function which is discontinuous at the point, all functions of which kind we have excluded from the present Treatise, it is unnecessary to give illustrative examples from other sources.

Ex. 4. Determine the nature of the point at the origin of the curve whose equation is

$$y^2(a^2 - x^2) = x^4,$$

$$\frac{dy}{dx} = \frac{2x^3 + xy^2}{y(a^2 - x^2)} = \frac{0}{0}, \text{ when } x = y = 0,$$

$$= \frac{6x^2 + 2xy\dfrac{dy}{dx} + y^2}{(a^2 - x^2)\dfrac{dy}{dx} - 2xy} = \frac{0}{a^2\dfrac{dy}{dx}}; \quad \ldots\ldots$$

$$\therefore \ \left(\frac{dy}{dx}\right)^2 = \frac{0}{a^2}, \qquad \therefore \ \frac{dy}{dx} = \pm 0;$$

that is, two branches of the curve pass through the origin and touch the axis of x on different sides of it, and are both in the plane of reference.

If the equation to the curve were

$$y^2(x^2 - a^2) = x^4,$$

the two branches which pass through the origin would touch the axis of x, but both would be out of the plane of reference; see fig. 77.

Discuss the nature of the point at the origin of
ose equation is

$$y^2 = ax^3,$$

$$\frac{dy}{dx} = \frac{3ax^2}{2y} = \frac{0}{0}, \text{ when } x = y = 0,$$

$$= \frac{6ax}{2\frac{dy}{dx}};$$

$$\therefore \quad \left(\frac{dy}{dx}\right)^2 = 3ax,$$

$$\frac{dy}{dx} = \pm (3ax)^{\frac{1}{2}} = 0, \text{ if } x = 0,$$

affected with $\pm$, when x is positive, but with $\pm \sqrt{-}$, if
nce at the origin there is a cusp, both branches
the axis of x, and the curve is out of the plane
en x is negative. The curve is, on account of
of its equation, called the Semicubical Parabola.
o since

$$\frac{d^2y}{dx^2} = \pm \frac{3a^{\frac{1}{2}}}{4x^{\frac{1}{2}}},$$

which is positive or negative according as the branch of the curve is above or below the axis of x, the cusp is of the first species; see fig. 64.

Ex. 6. Discuss the nature of the point at the origin of the curve

$$y^3 = ax^2 - x^3,$$

$$\frac{dy}{dx} = \frac{2ax - 3x^2}{3y^2} = \frac{0}{0}, \text{ at the origin,}$$

$$= \frac{2a - 6x}{6y\frac{dy}{dx}};$$

$$\therefore \quad \left(\frac{dy}{dx}\right)^2 = \frac{a}{3y},$$

$$\frac{dy}{dx} = \pm \left(\frac{a}{3y}\right)^{\frac{1}{2}} = \pm \infty, \text{ if } y = 0.$$

Hence the curve has two branches at the origin touching the axis of y, but as $\frac{dy}{dx}$ is affected with $\pm\sqrt{-}$, when y is negative, it shews that at the origin there is a cusp of the first species, and such as is drawn in fig. 65.

Ex. 7. Determine the nature of the point of the curve

$$y^2 = x(x+a)^2,$$

when $$x = -a, \quad \text{and } y = 0;$$

$$\frac{dy}{dx} = \frac{3x^2+4ax+a^2}{2y} = \frac{0}{0}, \text{ when } x = -a, \text{ and } y = 0,$$

$$= \frac{6x+4a}{2\frac{dy}{dx}};$$

$$\therefore \quad \left(\frac{dy}{dx}\right)^2 = -a, \quad \frac{dy}{dx} = \pm\sqrt{-a};$$

that is, there is a conjugate point, both branches of the curve being out of the plane of the paper, but piercing it at the point, and their directions making, with the axes of x, angles whose tangents are $\pm\sqrt{a}$.

205.] There is an explicit form to which the equations of many curves admit of being reduced, and which is well adapted to exhibit the peculiarities of cusps of both species.

Suppose that the equation to a curve can be put in the form

$$y = f(x) \pm \phi(x), \tag{72}$$

of which $f(x)$ is possible for all values of x through which we consider it, but that $\phi(x)$ is possible for some and impossible for others; and, to fix our thoughts, suppose that $\phi(x)$ is impossible or possible, according as x is less or greater than a. The curve whose equation is $y = f(x)$ is aptly called the *diametral* curve of (72), the ordinates of (72) being equal to $f(x)$ increased and diminished by the same quantity, viz. $\phi(x)$. Then, if $\phi(x)$ is such that $\phi(a) = 0$ and $\phi'(a) = 0$, the curve (72) has, when $x = a$, a common point, and is coincident in direction, with $y = f(x)$; but as two branches unite at the point, and are distinct when $x = a+h$, and affected

with $\pm\sqrt{-}$ when $x = a - h$, we have a cusp of the first or second species, according as the curvature is turned in opposite or in the same directions. The following examples will illustrate the method.

Ex. 1. $$(y - a - x)^2 = (x - c)^5,$$

$$y = a + x \pm (x - c)^{\frac{5}{2}},$$

$$\frac{dy}{dx} = 1 \pm \frac{5}{2}(x - c)^{\frac{3}{2}},$$

$$\frac{d^2y}{dx^2} = \pm\frac{15}{4}(x - c)^{\frac{1}{2}}.$$

Hence the diametral line is that whose equation is $y = a + x$, and as y is affected with $\pm\sqrt{-}$ for all values of x less than c, it follows that the curve is in or out of the plane of reference, according as x is greater or less than c; and as $\frac{dy}{dx} = 1$ when $x = c$, it is plain that at that point both branches touch a common tangent, one being above and the other below it, and as $\frac{d^2y}{dx^2}$ is $\pm$ when x is greater than c, the cusp is of the first species; see fig. 66.

Ex. 2. $$(y - x^2)^2 = ax^5,$$

$$y = x^2 \pm a^{\frac{1}{2}}x^{\frac{5}{2}},$$

$$\frac{dy}{dx} = 2x \pm \frac{5a^{\frac{1}{2}}}{2}x^{\frac{3}{2}},$$

$$\frac{d^2y}{dx^2} = 2 \pm \frac{15a^{\frac{1}{2}}}{4}x^{\frac{1}{2}}.$$

From the above equations it appears that the diametral curve is the parabola whose equation is $y = x^2$; that the curve is in the plane of reference on the positive side of the axis of y, and out of it on the negative side; that there is a cusp of the second species at the origin, since $\frac{d^2y}{dx^2}$ is positive for both branches; see fig. 67.

Ex. 3. $$y = a + bx + cx^2 \pm x^{\frac{3}{2}}.$$

$$\frac{dy}{dx} = b + 2cx \pm \frac{3}{2}x^{\frac{1}{2}},$$

$$\frac{d^2y}{dx^2} = 2c \pm \frac{3}{4x^{\frac{1}{2}}}.$$

The diametral curve is plainly a parabola, and the curve is in or out of the plane of reference, according as x is positive or negative; there is a cusp of the first species at the point where the diameter cuts the axis of y, and the tangent at the point makes, with the axis of x, an angle whose tangent is b; see fig. 68.

206.] Returning to equation (67), Art. 203, let us suppose that at the critical point under discussion

$$\left(\frac{d^2\text{F}}{dx^2}\right) = 0, \quad \left(\frac{d^2\text{F}}{dx\,dy}\right) = 0, \quad \left(\frac{d^2\text{F}}{dy^2}\right) = 0;$$

then the value of $\frac{dy}{dx}$ again assumes the form $\frac{0}{0}$, and the numerator and denominator of it must be differentiated; in which operation however it is to be borne in mind, that $\frac{dy}{dx}$ does not vary with x and y near to the critical point, and is therefore to be considered constant; the true meaning and effect of these successive differentiations being as follows. Several branches of the curve have certain consecutive points in common, and certain elements in common; whilst therefore we are considering the curve, as to its continuation at one or more of these common points, it is indeterminate to which branch of the curve the points and elements belong, and therefore we must pass on from these common points to those contiguous ones which are on different branches; in which case the tangent lines drawn through these become separate for each branch, and the direction of each thereby becomes determined. Let the reader try to draw for himself an infinitely-magnified diagram of such points and curves in the same manner as we have drawn fig. 56.

Differentiating therefore the numerator and denominator of

the right-hand member of equation (67), and dividing through by dx,

$$\frac{dy}{dx} = -\frac{\left(\frac{d^3 F}{dx^3}\right) + 2\left(\frac{d^3 F}{dx^2\,dy}\right)\frac{dy}{dx} + \left(\frac{d^3 F}{dx\,dy^2}\right)\frac{dy^2}{dx^2}}{\left(\frac{d^3 F}{dy^3}\right)\frac{dy^2}{dx^2} + 2\left(\frac{d^3 F}{dx\,dy^2}\right)\frac{dy}{dx} + \left(\frac{d^3 F}{dx^2\,dy}\right)}; \quad (73)$$

whence, multiplying and reducing,

$$\left(\frac{d^3 F}{dy^3}\right)\frac{dy^3}{dx^3} + 3\left(\frac{d^3 F}{dx\,dy^2}\right)\frac{dy^2}{dx^2} + 3\left(\frac{d^3 F}{dx^2\,dy}\right)\frac{dy}{dx} + \left(\frac{d^3 F}{dx^3}\right) = 0, \quad (74)$$

a cubic equation in $\frac{dy}{dx}$, and therefore with three roots, shewing that three branches of the curve pass through the critical point, which is accordingly called a *triple* point; the three branches being all in the plane of reference, or one in and two out of the plane, according as the roots of (74) are all *real*, or one *real* and two impossible.

As the criteria of this division lead to a long and complicated expression, it is needless to investigate it here; and moreover, as the determination of the several values of $\frac{dy}{dx}$, corresponding to the several branches of the curve, is not difficult, we shall only add an example.

Ex. 1. To determine the Nature of the Point at the Origin of the Curve whose Equation is

$$x^4 - ayx^2 + by^3 = 0.$$

$$\frac{dy}{dx} = \frac{4x^3 - 2axy}{ax^2 - 3by^2} = \frac{0}{0} \text{ at origin,}$$

$$= \frac{12x^2 - 2ax\frac{dy}{dx} - 2ay}{2ax - 6by\frac{dy}{dx}} = \frac{0}{0},$$

$$= \frac{24x - 4a\frac{dy}{dx}}{2a - 6b\frac{dy^2}{dx^2}} = \frac{-4a\frac{dy}{dx}}{2a - 6b\frac{dy^2}{dx^2}}, \text{ when } x = y = 0;$$

$$\therefore \quad 6a\frac{dy}{dx} - 6b\frac{dy^3}{dx^3} = 0:$$

whence $$\frac{dy}{dx} = 0, \quad \text{and} \quad \frac{dy}{dx} = \pm\left(\frac{a}{b}\right)^{\frac{1}{2}};$$

and therefore at the origin there is a triple point, as three branches of the curve pass through it, of which one touches the axis of x, and the other two are inclined to it at angles whose tangents are $\pm\left(\frac{a}{b}\right)^{\frac{1}{2}}$; see fig. 69.

207.] Similarly if all the third partial differential coefficients vanish at the point under discussion, we must differentiate again the numerator and denominator of the right-hand member of (73); by which means we shall obtain a biquadratic expression in $\frac{dy}{dx}$, indicating that four branches of the curve pass through the point, which is therefore called a *quadruple* point.

208.] Such is the general theory of multiple points; of which the analytical note is the vanishing at the point of successive partial differential coefficients of the implicit equation to the curve. That such must vanish, if many branches pass through the same point, may thus be shewn *a priori*.

Let a curve be such that, when $x = a \pm h$, y has many values, or, to borrow language from the theory of equations, the equation formed in powers of y has many unequal roots, but when $x = a$ several of these values of y become equal, say $y = b$; then, in this case, as many roots which before were unequal must become equal, as there are branches passing through the point; and thus there will be many equal factors multiplied together, which will produce a factor of the form $(y-b)^n$. By a similar train of reasoning we may prove that at such a point many factors involving x, which at other points are unequal, become equal; and we have a factor of the form $(x-a)^m$, m and n being some numerical quantities at least greater than 1. Now since differentiation diminishes the exponent of such a quantity only by unity, it is plain that $\left(\frac{dF}{dx}\right)$ will, at the point in question, have a factor of the form

$(x-a)^{m-1}$, and therefore will $= 0$. Similarly $\left(\frac{dF}{dy}\right)$ will have a factor of the form $(y-b)^{n-1}$, and will $= 0$ also; and according to the numerical magnitudes of m and n will be the number of branches passing through the point, and the number of successive partial differential coefficients which $= 0$, for the values $x = a$, $y = b$.

SECTION 5.—*On Tracing Curves by means of their Equations.*

209.] Our object in the present Section is to analyse the equation to a curve, and to trace the figure, so far as the results of algebraical geometry and of the discussion of the previous Articles of the present Chapter enable us to do. *All* curves we cannot trace, any more than find the equations to all figures of a character however complicated; the problem is as general as the solution of *all* equations; and therefore what follows is to be taken as an explanation, and as a specimen, of the slight means we possess of discussing some few simple cases.

1) If the equation admits of being simplified by a change of origin, or by turning the axes through any angle, or by transforming the equation into its equivalent in terms of polar coordinates, let such a change be effected before we begin the analysis. Thus, for instance, the equation $x^2-2ax+y^2+2by = 0$, admits of being discussed more easily when for x we write $x + a$, and for y, $y - b$: whereby the result becomes $x^2 + y^2 = a^2 + b^2$. Similarly the curve whose equation is $(x^2 + y^2)^{\frac{1}{2}} = a\tan^{-1}\left(\frac{y}{x}\right)$ is more easily traced when it is put in its equivalent polar form, $r = a\theta$; the means of tracing Polar Curves will be the subject of the next Chapter.

2) If the equation to a curve admits of being put in the form

$$y = f(x) \pm \phi(x),$$

in which case, as before observed, $y = f(x)$ is the equation to a curve diametral to the curve to be traced, it is most convenient first to trace the diametral curve, and then to increase and diminish its ordinate by the quantity $\phi(x)$ corresponding to the several values of the abscissa to the curve $y = f(x)$.

Thus, for instance, in the discussion of the general equation of the second degree,

$$ay^2 + bxy + cx^2 + dy + ex + f = 0, \tag{76}$$

let the equation be solved for y; whence

$$y = -\frac{b}{2a}x - \frac{d}{2a}$$
$$\pm \frac{1}{2a}\left\{(b^2 - 4ac)x^2 + 2(bd - 2ae)x + d^2 - 4af\right\}^{\frac{1}{2}}; \tag{77}$$

$y = -\frac{b}{2a}x - \frac{d}{2a}$, is the equation to a straight line, and therefore the ordinate to the curve is the ordinate to a straight line increased and diminished by equal quantities; the most convenient method therefore of tracing the curve is first to construct the straight line, and then to add to and subtract from its ordinate such a quantity as arises from an examination of the latter part of (77).

3) Let the equation to the curve, if possible, be put in the explicit form $y = f(x)$; and let all the points be determined at which the curve meets the coordinate axes, by finding the values of x which render $y = 0$, and the values of y which render $x = 0$; and let the change or continuation of sign be examined in order to determine whether the curve passes from below to above the axis of x, or *vice versâ*; and whether it passes from the left to the right of the axis of y, or *vice versâ*; or whether it touches the axes; and if it cuts the axis, let the value of $\frac{dy}{dx}$ be examined at the point of section, so as to determine the angle at which it cuts. And if, for all values of x from $+\infty$ to $-\infty$, y is unaffected with $\pm\sqrt{-}$, the curve extends infinitely in both directions in the plane of the paper; but if at any point, say $x = a$, $y = b$, the equation is such as on one side of that point to be affected with $\pm$, and on the other side with $\pm\sqrt{-}$, then at that point the curve leaves the plane of the paper. Suppose that at such a point there is only one branch of the curve, so that the symbol of "impossibility" does not arise from the extraction of the square root of a negative number, then there is what is by French writers termed a *point d'arret*; or as we may conveniently call it, a point of abrupt termination; and the branch has only one tangent.

Such however can only arise from a discontinuous function, or from such functions as those for which Maclaurin's Theorem fails. Thus, if the equation be

$$y = x^2 \log_e x,$$

$y = 0$, when $x = 0$, by virtue of Ex. 2, Art. 111; also $\frac{dy}{dx} = 0$, when $x = 0$; hence the curve passes through the origin, and touches the axis of x, and is in the plane of the paper on the positive side of the axis of y; but as the logarithms of negative numbers are (see Art. 62) affected with $\pm \sqrt{-}$, the curve is out of the plane of reference on the negative side of the same axis; and therefore there is at the origin a point of abrupt termination. The above curve is traced in fig. 70 as far as it exists in the plane of the paper; AO $= 1$.

If however at the point where the ordinate becomes affected with $\pm \sqrt{-}$ two branches pass into another plane, there is either a cusp or a salient point, according as the two branches have the same or different tangents. The distinctive characters of these points depend on the corresponding value or values of $\frac{dy}{dx}$. And if the equation to the curve is satisfied by $x = a$, $y = b$; but when x is increased or decreased by a quantity, however small, y is affected with $\pm \sqrt{-}$, then at such a point the curve, which lies in some other plane, pierces the plane of reference; and the point is a conjugate or isolated point; and of course one or two or more branches of a curve may pass through such a point: as for instance if the equation to a line be

$$y - b = (-)^{\frac{1}{2}} (x - a),$$

the equation is satisfied by $x = a$, $y = b$, which indicates a point in the plane of reference; but every other point of the line is in the plane passing through the line BD (see fig. 71, OA $= a$, AB $= b$), and perpendicular to the plane of the paper.

When two branches of the curve simultaneously pierce the plane of the paper, the two roots of (68), Art. 203, are impossible, as is the case in Ex. 8, which is traced below in Art. 211. And a curve may have any number of such conjugate points, by continually passing through the plane of the paper, such as in the subjoined example:

$$y = ax^2 \pm (bx)^{\frac{1}{2}} \sin x.$$

The curve is traced in fig. 72, the dotted line indicating the branches in a plane perpendicular to that of the paper. $y = ax^2$, which represents the diametral curve, is a parabola, B′OB, drawn in the figure; and the ordinate to the curve is periodically reduced to its ordinate when $x = 0$, or $= \pi$, or $= 2\pi$, ... or $=$ any multiple of π; but when x is negative, the part of the ordinate to be added or subtracted to the ordinate of the parabola is affected with $(-)^{\frac{1}{2}}$, except at the points where $x =$ some multiple of π, at which the branches of the curve pierce the plane of reference: and thus it continues *ad infinitum*, the curve itself being continuous, but there being a series of discontinuous points, if we consider only those points which the plane of the paper contains.

Curves such as the last have been called "Courbes Pointillés*;" which name however has been given by writers who discard the mode of interpretation of the symbol of impossibility which we have employed in this Treatise, and are therefore obliged to allow that algebraical expressions admit of discontinuous *geometrical* interpretation: a result surely utterly at variance with the *algebraical* nature of such functions, which admit of differentiation, and thereby indicate that they fulfil the law of continuity.

4) On the method of determining the increase and decrease of x and y nothing more need be said; but we must be careful to investigate the points at which $\frac{dy}{dx} = 0$, and $= \infty$, and to observe whether or not there is a change of sign, as such is the criterion of maxima and minima. With this object we shall equate $\frac{dy}{dx}$ to 0 and ∞, and examine the course the curve takes at these critical points.

5) In regard to asymptotes, and the course of the curve with respect to them, we must examine for what finite values of x, y is infinite, and for what values of y, x is infinite, as such will be asymptotes parallel to the axes of y and x respectively; and by investigating whether $\frac{dy}{dx}$ changes sign or not for these

* See page 382 of a "Treatise on the Differential Calculus," by Augustus De Morgan, M.A. Baldwin and Cradock, London, 1842.

asymptotic values, we shall determine whether the infinite ordinate is a maximum or minimum; that is, whether it returns, or whether it continues round the circle of infinite radius, such as we described in the last Chapter; and which of the forms delineated in figs. 24, 25, 26, 51 the curve takes. We must also be careful to determine whether there are rectilineal asymptotes inclined at oblique angles to the axes of coordinates, and whether the curve be above them or below them. It may happen that two distinct branches of a curve will approach the same asymptote. Sometimes also a curve will cut its asymptote; as e.g. if $y = a\,\dfrac{\sin x}{x}$, the axis of x is an asymptote, and the curve cuts it whenever $x =$ a multiple of π.

6) The general character of a curve, with regard to the curvature of it in a particular direction and its points of inflexion, has been sufficiently discussed above in Section 3 of the present Chapter. Practically however $\dfrac{d^2y}{dx^2}$ is of little use in enabling us to trace a curve, unless it assumes a simple and explicit form; and should also at any point of the curve $\dfrac{d^2y}{dx^2} = 0$, and not change its sign, we may conclude that more than two elements of the curve are in one and the same straight line. (See Art. 202.)

7) There is nothing more to be added on the theory of multiple points and their varieties.

8) And generally it is of little use to examine the values of x and y, except at such critical points as we have above described; and except when $x = \infty$, and $y = \infty$, in order that we may determine the course the curve is taking at such distances from the axes.

210.] In the discussion of any particular equation representing a plane curve, the method indicated by the following rules is the most convenient to adopt:

I. Reduce the equation if possible to the explicit form, and simplify it, as far as may be, by means of a change of origin, or by a transformation into polar coordinates.

II. Discover, arrange, and tabulate with their proper signs, all the critical values of y and x, both in and out of the plane of reference.

III. Discuss and tabulate the critical values of $\frac{dy}{dx}$, as e.g. determine at what angles the curve cuts the axis, the maximum and minimum ordinates, &c.

IV. Find the equations to the asymptotes, and determine whether the curve is above or below them.

V. Find (if it be possible in a convenient form) $\frac{d^2y}{dx^2}$; thence determine the direction of curvature, and points of inflexion.

VI. If at any point $\frac{dy}{dx} = \frac{0}{0}$, evaluate the quantity, and determine the several double, triple points, &c.

211.] Examples illustrative of the preceding theory.

Ex. 1. Trace the Curve whose Equation is $y^2 = a^2 x^3$.

$$\therefore \quad y = \pm a x^{\frac{3}{2}}, \tag{78}$$

$$\frac{dy}{dx} = \frac{3a^2x^2}{2y}, \text{ also } \frac{dy}{dx} = \pm \frac{3ax^{\frac{1}{2}}}{2},$$

$$\frac{d^2y}{dx^2} = \pm \frac{3a}{4x^{\frac{1}{2}}}.$$

From (78) it is plain that the curve is symmetrical with respect to the axis of x, and since the curve passes through the origin, the first value of $\frac{dy}{dx}$ at that point assumes an indeterminate form, the value of which, as before shewn in Ex. 5, Art. 204, is such as to give a cusp, both branches of which touch the axis of x, and which are in the plane of the paper on the positive side of the axis of y, and out of it on the negative side. The same thing is also apparent from the second value of $\frac{dy}{dx}$, which $= 0$ at the origin, and is affected with $\pm\sqrt{-}$ when x is negative, and with $\pm$ when x is positive; also $\frac{d^2y}{dx^2}$, being affected with $\pm$, shews that one branch of the curve is convex, and the other concave, downwards.

Hence we may tabulate as follows:

	x	y	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$
1	0	$\pm\sqrt{-}, 0, \pm$	$\pm\sqrt{-}, 0, \pm$	$\pm\sqrt{-}, \infty, \pm$
2	$+\infty$	$\pm\infty$	$\pm\infty$	$\pm$
3	$-\infty$	$\pm\sqrt{-}\,\infty$	$\pm\sqrt{-}\,\infty$	$\pm\sqrt{-}$

From 1 it appears that the curve passes through the origin with two branches, both touching the axis of x, one of which is convex, and the other is concave, downwards, and which are out of the plane of the paper on the negative side of the axis of x, and are in it on the positive side. From 2 and 3 it appears, that as x increases, whether positively or negatively, y increases also, and since $\frac{dy}{dx}$ approximates to ∞, that the curve approaches to parallelism with the axis of y; the only critical value is $x = 0$; the curve is drawn in fig. 64.

Ex. 2. Discuss the Curve

$$y = x^3 - 2x^2 - 5x + 6,$$

$$= (x+2)(x-1)(x-3),$$

$$\frac{dy}{dx} = 3x^2 - 4x - 5,$$

$$= 3\left(x - \frac{2+\sqrt{19}}{3}\right)\left(x - \frac{2-\sqrt{19}}{3}\right),$$

$$\frac{d^2y}{dx^2} = 6x - 4,$$

$$= 6\left(x - \frac{2}{3}\right).$$

As the equation does not admit of expansion in descending powers of x of a form, such that the highest positive power of x may be unity, it follows that the curve has no rectilineal asymptote.

	x	y	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$
1	-2	$-, 0, +$	$+15$	$-$
2	0	$+6$	-5	$-$
3	1	$+, 0, -$	-6	$+$
4	3	$-, 0, +$	$+10$	$+$
5	$\frac{2+\sqrt{19}}{3}$	$-$	$-, 0, +$	$+$
6	$\frac{2-\sqrt{19}}{3}$	$+$	$+, 0, -$	$-$
7	$\frac{2}{3}$	$+$	$-$	$-, 0, +$
8	$+\infty$	$+\infty$	$+\infty$	$+$
9	$-\infty$	$-\infty$	$+\infty$	$-$

On examination of which table it appears that, when $= -\infty$, the curve is at an infinite distance below the axis of , approaching to parallelism with the axis of y, and being concave downwards; whence it cuts the axis of x, when $x = -2$, s shewn by 1) at a large acute angle, and the ordinate attains maximum at the value of x given by 6; whence the ordinate ecreases, cutting the axis of y at a distance $+6$ from the rigin, and being concave downwards until $x = \frac{2}{3}$, at which oint, as shewn by 7, there is a point of inflexion; and the urve being convex downwards cuts the axis of x, when $x = 1$, nd decreases until x is equal to the value given in 5, at which oint there is a minimum ordinate; after which the ordinate gain increases, cuts the axis of x when $x = 3$, and goes off to n infinite distance approaching to parallelism with the axis of , and being convex downwards. The curve is drawn in fig. 73, A $= 2$, OB $= 1$, OC $= 3$; OE $= \frac{2-\sqrt{19}}{3}$, OD $= \frac{2+\sqrt{19}}{3}$, F $= \frac{2}{3}$.

Discuss the Curve whose Equation is

$$\frac{x^3}{a^3} + \frac{y^3}{b^3} = 1,$$

$$y = \frac{b}{a}(a^3 - x^3)^{\frac{1}{3}},$$

$$\frac{dy}{dx} = -\frac{b}{a}\frac{x^2}{(a^3 - x^3)^{\frac{2}{3}}}; \text{ also } \frac{dy}{dx} = -\frac{b^3}{a^3}\frac{x^2}{y^2},$$

$$\frac{d^2y}{dx^2} = \frac{-2a^2bx}{(a^3 - x^3)^{\frac{5}{3}}}.$$

asymptote,

$$y = -\frac{b}{a}x\left(1 - \frac{a^3}{x^3}\right)^{\frac{1}{3}},$$

$$= -\frac{b}{a}x\left\{1 - \frac{a^3}{3x^3} - \dots\dots\right\};$$

e the equation to the asymptote is

$$y = -\frac{b}{a}x,$$

and as the next term of the expansion is positive, the curve lies above the asymptote.

	x	y	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$
1	0	b	$-, 0, -$	$+, 0, -$
2	a	$+, 0, -$	$-, \infty, -$	$-, \infty, +$
3	$+\infty$	$-\infty$	$-\frac{b}{a}$	$+$
4	$-\infty$	$+\infty$	$-\frac{b}{a}$	$+$

An inspection of which table, together with the equation to the asymptote, shews that the curve is such as that drawn in fig. 74 (OA $= a$, OB $= b$), with points of inflexion at A and B.

Ex. 4. **Discuss the Curve whose Equation is**

$$y = \frac{x}{1+x^2},$$

$$\frac{dy}{dx} = \frac{1-x^2}{(1+x^2)^2},$$

$$\frac{d^2y}{dx^2} = \frac{2x(x^2-3)}{(1+x^2)^3}.$$

	x	y	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$
1	0	$-, 0, +$	1	$+, 0, -$
2	$+1$	$\frac{1}{2}$	$+, 0, -$	$-$
3	-1	$-\frac{1}{2}$	$-, 0, +$	$+$
4	$+\sqrt{3}$	$+$	$-$	$-, 0, +$
5	$-\sqrt{3}$	$-$	$-$	$-, 0, +$
6	$+\infty$	$+0$	0	$+$
7	$-\infty$	-0	0	$-$

An inspection of which table shews, that the curve is such as s drawn in fig. 75. For when $x = -\infty$, $y = -0$, that is, he curve lies immediately below the axis of x, which is an symptote to it; and the curve recedes further from it on the egative side, and when $x = -\sqrt{3} = $ OB′, there is a point of nflexion; for the curve, having been concave, becomes conex downwards; and when $x = -1 = $ OA′, the curve is at its reatest distance below the axis of x, and there is a minimum rdinate; after which the curve approaches to the axis of x, nd passes through the origin, cutting the axis of x at 45°, and oes through the same phases as on the negative side, except hat it is above instead of being below the axis of x; and as he sign of y changes from $+0$ to -0, when x changes from $+\infty$ to $-\infty$, so do the two branches which are asymptotic to he axis of x unite, and the curve is continuous from $+\infty$ to $-\infty$; see Art. 159, 160.

Discuss the Curve whose Equation is

$$y^3 = 2ax^2 - x^3,$$

$$y = x^{\frac{2}{3}}(2a - x)^{\frac{1}{3}},$$

$$\frac{dy}{dx} = \frac{4a - 3x}{3x^{\frac{1}{3}}(2a - x)^{\frac{2}{3}}}; \text{ also } \frac{dy}{dx} = \frac{4ax - 3x^2}{3y^2},$$

$$\frac{d^2y}{dx^2} = \frac{-8a^2}{9x^{\frac{4}{3}}(2a - x)^{\frac{5}{3}}}.$$

the equation to the asymptote,

$$y = -x\left(1 - \frac{2a}{x}\right)^{\frac{1}{3}},$$

$$= -x\left\{1 - \frac{2a}{3x} - \dots\dots\right\};$$

. $y = -x + \dfrac{2a}{3}$, is the equation to the asymptote, and, as the next term of the expansion is positive, the curve lies above the asymptote.

Also since $x = 0$, if $y = 0$, and in this case $\dfrac{dy}{dx} = \dfrac{0}{0}$, it must be evaluated;

$$\frac{dy}{dx} = \frac{4ax - 3x^2}{3y^2} = \frac{0}{0}, \text{ when } x = y = 0,$$

$$= \frac{4a - 6x}{6y\dfrac{dy}{dx}};$$

$$\therefore \left(\frac{dy}{dx}\right)^2 = \frac{2a}{3y} = \infty, \text{ when } x = y = 0,$$

$$\therefore \frac{dy}{dx} = \pm\left(\frac{2a}{3y}\right)^{\frac{1}{2}} = \pm\infty.$$

Therefore there are at the origin two branches of the curve touching the axis of y; and the value of $\dfrac{dy}{dx}$ shews that, if y is negative, $\dfrac{dy}{dx}$ is affected with $\pm\sqrt{-}$, and therefore the origin is a cusp of the first series.

The existence also of such a cusp is manifest from the explicit value of $\frac{dy}{dx}$; hence the table is as follows:

	x	y	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$
1	0	$+, 0, +$	$\frac{0}{0} = \pm\infty$	$-$
2	$2a$	$+, 0, -$	$-, \infty, -$	$-, \infty, +$
3	$\frac{4a}{3}$	$+$	$+, 0, -$	$-$
4	$+\infty$	$-\infty$	-1	$+, 0, -$
5	$-\infty$	$+\infty$	-1	$-, 0, +$

Hence, and from the asymptote, the curve is that delineated in fig. 76, in which OA $= 2a$, OB $= \frac{4a}{3}$, OC $= \frac{2a}{3}$. For it appears from 1 that the curve passes through the origin, which is a cusp of the first species, the two branches touching the axis of y, and above the axis of x, both branches being concave downwards; and the curve, having been above the axis of x, from $x = 0$ to $x = 2a$, at this last point cuts the axis of x at right angles, and changes its curvature: for having been concave, it becomes convex downwards. 3 shews that the curve has attained to a maximum ordinate when $x = \frac{4a}{3}$; the curve approaches to the asymptote whose equation is $y = -x + \frac{2a}{3}$, which, as is shewn by 4 and 5, it cuts and touches when $x = \infty$, where is a point of inflexion, and thus the two asymptotic branches unite. We have traced the curve only in the plane of reference, as we have not discussed the geometrical meaning of the cube roots of $+$.

Ex. 6. Discuss the Cissoid of Diocles, the equation to which is

$$y^2 = \frac{x^3}{2a - x};$$

$$\therefore \quad y = \pm \frac{x^{\frac{3}{2}}}{(2a - x)^{\frac{1}{2}}}.$$

ected with $\pm$, the curve is symmetrical with re- … axis of x.

$$\frac{dy}{dx} = \pm \frac{x^{\frac{1}{2}}(3a - x)}{(2a - x)^{\frac{3}{2}}}; \quad \text{also } \frac{dy}{dx} = \frac{x^2(3a - x)}{y(2a - x)^2},$$

$$\frac{d^2y}{dx^2} = \pm \frac{3a^2}{x^{\frac{1}{2}}(2a - x)^{\frac{5}{2}}}.$$

the equations to the asymptotes as found in Ex. 2, are

$$y = \pm \sqrt{-}\,(x + a).$$

nce the table becomes

	x	y	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$
	0	$\pm \sqrt{-}, 0, \pm$	$\pm \sqrt{-}, 0, \pm$	$\pm \sqrt{-}, \infty, \pm$
	$2a$	$\pm, \infty, \pm \sqrt{-}$	$\pm, \infty, \pm \sqrt{-}$	$\pm, \infty, \pm \sqrt{-}$
3	$3a$	$\pm \sqrt{-}$	$\pm \sqrt{-}, 0, \pm \sqrt{-}$	$\pm \sqrt{-}$
4	a	$\pm a$	± 2	$\pm$
5	$+\infty$	$\pm \sqrt{-}\,\infty$	$\pm \sqrt{-}$	$\pm \sqrt{-}$
6	$-\infty$	$\pm \sqrt{-}\,\infty$	$\pm \sqrt{-}$	$\pm \sqrt{-}$

Hence, and by means of the asymptote, the curve is that delineated in fig. 34; OC = CA = a; OB = $3a$, OB being the abscissa corresponding to the maximum and minimum ordinates of the curve out of the plane of the paper.

Ex. 7. Discuss the Witch of Agnesi, the equation to which is

$$y^2 = 4a^2\frac{2a - x}{x},$$

$$y = \pm 2a\left(\frac{2a - x}{x}\right)^{\frac{1}{2}}.$$

Thus it appears that the curve is symmetrical with respect to the axis of x;

$$\frac{dy}{dx} = \mp \frac{2a^2}{x^{\frac{1}{2}}(2a-x)^{\frac{3}{2}}}; \quad \text{also } \frac{dy}{dx} = -\frac{4a^3}{x^2 y},$$

$$\frac{d^2y}{dx^2} = \pm \frac{2a^2(3a-2x)}{x(2ax-x^2)^{\frac{3}{2}}}.$$

Also the equations to the asymptotes as found in Ex. 3, Art. 193, are

$$y = \pm \sqrt{-}\, 2a;$$

that is, the asymptotes are two straight lines out of the plane of the paper, parallel to the axis of x, and at distances $\pm 2a$ from it.

The table is as follows:

	x	y	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$
1	0	$\pm\sqrt{-}, \infty, \pm$	$\pm\sqrt{-}, \infty, \pm$	$\pm\sqrt{-}, \infty, \pm$
2	$2a$	$\pm, 0, \pm\sqrt{-}$	$\pm, \infty, \pm\sqrt{-}$	$\pm$
3	$\frac{3a}{2}$	$\pm$	$\pm$	$\pm, 0, \mp$
4	$+\infty$	$\pm\sqrt{-}\,2a$	$\pm\sqrt{-}\,0$	$\pm\sqrt{-}$
5	$-\infty$	$\pm\sqrt{-}\,2a$	$\pm\sqrt{-}\,0$	$\pm\sqrt{-}$

An examination of which table shews that the curve is that drawn in fig. 35, where OC = CA = a; OB = OB$'$ = $2a$.

Ex. 8. Discuss the Curve whose Equation is

$$y^2(x^2-a^2) = x^4;$$

$$\therefore \quad y = \pm \frac{x^2}{(x^2-a^2)^{\frac{1}{2}}}.$$

Since the given equation is not changed when we write $-x$ and $-y$ for $+x$ and $+y$ respectively, it appears that the curve is situated symmetrically in the four quadrants. Differentiating, we have

$$\frac{dy}{dx} = \pm \frac{x(x^2-2a^2)}{(x^2-a^2)^{\frac{3}{2}}}; \quad \text{also } \frac{dy}{dx} = \frac{2x^3-xy^2}{y(x^2-a^2)};$$

$$\frac{d^2y}{dx^2} = \pm \frac{a^2(x^2+2a^2)}{(x^2-a^2)^{\frac{5}{2}}}.$$

equations to the asymptote,

$$\pm \frac{x^2}{x\left(1-\frac{a^2}{x^2}\right)^{\frac{1}{2}}} = \pm x\left(1-\frac{a^2}{x^2}\right)^{-\frac{1}{2}},$$

$$\pm x\left\{1+\frac{a^2}{2x^2}+\ldots\ldots\right\},$$

$$= \pm\left\{x+\frac{a^2}{2x}+\ldots\ldots\right\};$$

the equations to the a totes are

$$y = \pm$$

1, as the sign of the next ter is positive, the curve lies ve the asymptote in the first rant.

When $x = 0$, $y = 0$; theref the curve passes through e origin, at which point $\frac{dy}{dx} =$ s appears from its second value given above, and has therefore to be evaluated.

$$\frac{dy}{dx} = \frac{2x^3 - xy^2}{y(x^2-a^2)} = \frac{0}{0}, \text{ when } x = y = 0,$$

$$= \frac{6x^2 - y^2 - 2xy\dfrac{dy}{dx}}{(x^2-a^2)\dfrac{dy}{dx} + 2xy}, \quad . \quad . \quad . \quad .$$

$$= \frac{0}{-a^2\dfrac{dy}{dx}}, \text{ when } x = y = 0;$$

$$\therefore \quad \text{at the origin } \left(\frac{dy}{dx}\right)^2 = -\frac{0}{a^2},$$

$$\frac{dy}{dx} = \pm\left(\frac{-0}{a^2}\right)^{\frac{1}{2}};$$

which implies that two branches of the curve touch the axis of x at the origin, both of which are out of the plane of the paper.

The table of critical values is as follows:

	x	y	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$
1	0	$\pm\sqrt{-}, 0, \pm\sqrt{-}$	$\pm\sqrt{-}, 0, \pm\sqrt{-}$	$\pm\sqrt{-}$
2	$+a$	$\pm\sqrt{-}, \infty, \pm$	$\pm\sqrt{-}, \infty, \pm$	$\pm$
3	$-a$	$\pm, \infty, \pm\sqrt{-}$	$\pm, \infty, \pm\sqrt{-}$	$\pm$
4	$a\sqrt{2}$	$\pm 2a$	$\mp, 0, \pm$	$\pm$
5	$-a\sqrt{2}$	$\pm 2a$	$\pm, 0, \mp$	$\pm$
6	$+\infty$	$\pm\infty$	± 1	$\pm, 0, \mp$
7	$-\infty$	$\pm\infty$	± 1	$\pm, 0, \mp$

From 1 it appears that the curve passes through the origin, and has two branches, both of which are out of the plane of the paper, and which touch the axis of x; whence, as 2 and 3 shew the curve recedes from the axis of x, until when $x = \pm a = \text{OA} = \text{OA}'$, $y = \pm\infty$, and there are two asymptotes parallel to the axis of y. For values of x outside of these lines, the curve is in the plane of reference, and returns towards the axis of x, until the ordinate reaches minimum and maximum values when $x = a\sqrt{2}$, as is shewn by 4 and 5, whence it recedes again towards the asymptotes whose equations are $y = \pm x$, and intersects them at ∞ in a point of inflexion, as shewn by 6 and 7, the curve lying above the asymptote in the first quadrant, and being symmetrically situated in the others. Its course is traced in fig. 77, where $\text{OA} = a$, $\text{OB} = \sqrt{2}a$, $\text{BC} = 2a$, and where the dotted line represents the curve out of the plane of reference.

If the equation to be discussed had been

$$y^2(a^2 - x^2) = x^4,$$

the branches of the curve which are in the plane of reference would have been out of it, and *vice versâ*. The continuity of curve is remarkable in both cases.

Ex. 9. **Discuss the Curve whose Equation is**

$$y^2 = x^2 \frac{a+x}{a-x},$$

$$y = \pm x \left(\frac{a+x}{a-x}\right)^{\frac{1}{2}};$$

whence it appears that the curve is symmetrical with respect to the axis of x;

$$\frac{dy}{dx} = \pm \frac{a^2 + ax - x^2}{(a+x)^{\frac{1}{2}}(a-x)^{\frac{3}{2}}},$$

$$\frac{d^2y}{dx^2} = \pm \frac{a^2(2a+x)}{(a+x)^{\frac{3}{2}}(a-x)^{\frac{5}{2}}};$$

and the equations to the asymptotes are

$$y = \pm \sqrt{-}\,(x+a).$$

Also since $\frac{dy}{dx} = 0$, when $x = \frac{a}{2}(1 \pm \sqrt{5})$, a careful inspection of the above quantities shews that the form of the curve is that drawn in fig. 78, the dotted branches being those out of the plane of the paper; OA = OB = BC = a,

$$\text{OF} = \frac{a}{2}\{1 + \sqrt{5}\}, \quad \text{OE} = \frac{a}{2}\{1 - \sqrt{5}\}.$$

Ex. 10. Examine the Folium of Des Cartes, the Equation to which is

$$y^3 - 3axy + x^3 = 0.$$

As shewn above in Ex. 3, Art. 195, the equation to the asymptote is

$$y = -x - a,$$

and at the origin there is a double point as shewn in Ex. 3, Art. 204.

Also since

$$\frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax},$$

$\frac{dy}{dx} = 0$, if $ay = x^2$; that is, if $x = a(2)^{\frac{1}{3}}$, and $y = a(4)^{\frac{1}{3}}$.

Also $\frac{dy}{dx} = \infty$, if $ax = y^2$; that is, if $x = a(4)^{\frac{1}{3}}$, and $y = a(2)^{\frac{1}{3}}$, the curve does not extend beyond these limits, and is such as is delineated in fig. 63.

Ex. 11. Trace the Curve $y = \sin x$.

$$\frac{dy}{dx} = \cos x, \quad \frac{d^2y}{dx^2} = -\sin x.$$

In tracing curves of this kind involving circular functions, the arc, of which the trigonometrical function is given, is to be measured along the coordinate axis; in the present case along axis of x, since $\sin x$ is involved in the equation, and the ordinates are to be constructed corresponding to the arcs or abscissæ thus measured; π, we must remember, is the symbol for the arithmetical number 3.14159; and we must give to x such values as will render y a quantity capable of construction. Thus, in the equations above, let

$$x = \frac{\pi}{2} = 1.57079\ldots;$$

$$y = \sin x = 1, \quad \frac{dy}{dx} = 0, \quad \frac{d^2y}{dx^2} = -1.$$

Hence we may form the following table:

	x	y	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$
1	0	$-, 0, +$	1	$+, 0, -$
2	$\frac{\pi}{2}$	1	$+, 0, -$	$-$
3	π	$+, 0, -$	-1	$-, 0, +$
4	$\frac{3\pi}{2}$	-1	$-, 0, +$	$+$
5	π	$-, 0, +$	1	$+, 0, -$

After which the values recur, and so on continually; hence there is a series of equal curves, such as are delineated in fig. 79, and extending to infinity in both the positive and negative directions.

Ex. 12. Trace the Curve whose Equation is $y = e^{\sec x}$, e being the base of the Napierian logarithms.

$$\frac{dy}{dx} = e^{\sec x} \sec x \tan x,$$

$$\frac{d^2y}{dx^2} = e^{\sec x} \sec x \left\{(\sec x)^3 + 2(\sec x)^2 - \sec x - 1\right\}.$$

Hence the table of critical values is as follows; h is a small increment of x.

	x	y	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$
1	0	e	$-, 0, +$	$+$
2	$\frac{\pi}{2} - h$	$+\infty$	$+\infty$	$+$
3	$\frac{\pi}{2} + h$	0	0	$+$
4	π	$\frac{1}{e}$	$+, 0, -$	$-$
5	$\frac{3\pi}{2} - h$	0	0	$+$
6	$\frac{3\pi}{2} + h$	$+\infty$	∞	$+$
7	2π	e	$-, 0, +$	$+$

Accordingly the curve is that drawn in fig. 80, where OA = AB = BC = CD = DE = $\frac{\pi}{2}$, OF = e, BL = $\frac{1}{e}$; the curve cuts the axis of y at a point where the ordinate is a minimum, and thence the ordinate increases until $x = \frac{\pi}{2}$, at which it is infinite; and immediately afterwards it is zero, so that at A there is a point of abrupt termination; whence the ordinate again increases and becomes a maximum when $x = \pi$, and decreases until $x = \frac{3\pi}{2}$, at which value there is another point of abrupt termination; and so on, as in the figure, the curve extends to infinity in both directions. There are also two points of inflexion between A and C.

CHAPTER XI.

ON PROPERTIES OF PLANE CURVES, AS DEFINED BY EQUATIONS REFERRED TO POLAR COORDINATES.

SECTION 1.—*On the Mode of Interpretation, and on the Equations, of Curves referred to Polar Coordinates.*

212.] IN the present Chapter we shall investigate, for polar curves, formulæ somewhat analogous to those which we have in the last Chapter discussed for curves referred to rectangular coordinates; but previously it is necessary to extend the usual mode of interpreting polar equations, so as to accommodate them in a greater degree to the law of continuity.

Let $r = f(\theta)$ be the equation to the curve. Then, taking a fixed point S as origin, which is called the *pole*, and a fixed line SX passing through it as the line of origination, which is called the *prime radius* (see fig. 81), it is manifest that the moveable radius, which is symbolized by r, may revolve about S in two directions; and thus, if the only datum be that r makes an angle θ with the prime radius, it is undetermined whether r is above or below SX: that is, whether r revolves *up* from SX from right to left, or *down* from left to right. Hence arises the necessity of some symbol of the direction in which r turns, so that angles formed in one direction may be differently symbolized to those formed in another. This indefiniteness will be avoided if we call angles positive when measured *up* from SX, as in fig. 81: that is, when the radius vector revolves round S in the direction indicated by the curved arrow; and negative when they are measured *down* from SX, and the radius vector revolves in the direction indicated by the curved arrow in fig. 82. In this case then, + and −, as affecting angles, indicate the two different directions in which r can revolve in the plane of the paper.

Again, suppose that for a given value of θ, r is affected with a negative sign, a question arises, in what direction is the negative r to be measured? No doubt, if r is affected with a positive sign, the length of it, determined by the equation to the curve, is to be measured from the pole along the revolving radius vector which is inclined at the given angle to the prime radius; as e.g. if a polar equation between r and θ is such that, when $\theta = \frac{\pi}{4}$, $r = a$, then a length $= a$ is to be measured from the pole along the revolving radius, which is inclined at 45° to the prime radius. From analogy therefore to what has been said in Art. 162 on the signs + and −, − r must be measured along the radius vector produced backwards; i.e. if, when $\theta = \frac{\pi}{4}$, $r = -a$, a line equal to a must be measured from the pole along the revolving radius produced backwards: that is, in a direction making an angle of 225° with the prime radius. In order the better to avoid confusion on this subject, conceive the revolving radius to be an arrow of variable length, such as we have drawn in figs. 81 and 82, the pole being a fixed point in it; then, if θ be the angle between the prime radius and the part of the arrow towards the barbed end, lines measured from S in the direction SP will be positive, and in the direction SQ negative. If therefore r is affected with a positive sign, it is to be measured towards the barbed end, but if with a negative sign, towards the feathered end of the arrow. In the figures different positions of the arrow are drawn, to indicate different positive and negative directions of r.

In the following Chapter we shall omit those particular values of r which are affected with $\pm\sqrt{-}$, as no satisfactory interpretation of such symbols in such a relation exists, and we shall consider those only which are affected with $\pm$; being careful however to make r revolve in both the positive and negative directions, otherwise at certain points the curve will appear to be discontinuous.

And for the purpose of illustration in the sequel, we must here insert an account of the mode of description, and the equations of some polar curves, many of which, having been treated of at length by old geometricians, possess no small historical interest.

213.] The Spiral of Archimedes.

Def.—If the length of the radius vector of a spiral is proportional to the angle through which it has moved from its originating position, the locus of its extremity is the Spiral of Archimedes.

Let a = the length of the radius, when the angle described is equal to unity*; and let r be its length after describing the angle θ; therefore the equation is

$$r = a\theta; \tag{1}$$

see fig. 83.

The curve therefore starts from the pole; and the radius vector, which at the beginning is equal to zero, $=$ SA $= a$, when it has revolved through the angle ASX, $=$ the unit angle; and at the end of the first complete revolution, is equal to $2\pi a$; and this is the distance between the points at which any radius vector is cut by two successive convolutions of the curve. The dotted curve is that described by the generating point, as the radius vector revolves in the negative direction.

214.] The Reciprocal Spiral.

The reciprocal or hyperbolic spiral is so called from the form of its equation, which is

$$r = \frac{a}{\theta}, \tag{2}$$

the form of the curve expressed by which is given in fig. 84. The radius vector $= \infty$ when $\theta = 0$, and the curve is asymptotic to the straight line B′AB, as will be shewn in the sequel. Also, when $\theta = 1 =$ A′SX, $r = a =$ SA′; also $r = 0$ when $\theta = \infty$; therefore, after an infinite number of revolutions, the curve falls into the pole. The curve has also the dotted branch arising from the revolution of r in the negative direction.

215.] The Lituus.

This spiral is so called from its form as delineated in fig. 85. Its equation is

$$r = \frac{a}{\theta^{\frac{1}{2}}}. \tag{3}$$

* The unit angle is that whose subtending arc is equal to the radius, and expressed in degrees = 57·29578. See Ex. 5, Art. 24.

The prime radius is an asymptote to the curve; which has a point of inflexion when $r = \text{SB} = a\sqrt{2}$, as will be shewn hereafter. Also, when $\theta = 1 = \text{ASX}$, $r = \text{SA} = a$; there is an apparent discontinuity at the pole and at the extremity of the infinite branch, which arises from our not interpreting r when affected with $\pm\sqrt{-}$, as such it will be if the radius vector be made to revolve in a negative direction.

216.] The Logarithmic Spiral.

DEF.—The logarithmic spiral is that whose radius vector increases in a geometric, as its angle increases in an arithmetic ratio.

Hence the equation is $$r = a^{\theta}. \tag{4}$$

Therefore when $\theta = 0$, $r = 1 = \text{SA}$; when $\theta = 1$, $r = a$; when $\theta = \infty$, $r = \infty$; when $\theta = -\infty$, $r = 0$; and therefore the spiral runs into its pole after an infinite number of revolutions in the negative direction; the spiral is also called the Equiangular Spiral from a property which will subsequently be proved, viz. that it cuts all its radii vectores at a constant angle; that is, the angle SPT is constant, at whatever point P be. Its form is delineated in fig. 86.

217.] The Involute of the Circle; fig. 87.

DEF.—The involute of the circle is the curve formed by the extremity of an inextensible string, as it is wrapped round the circumference of a circle.

If r be the radius vector of a curve, and p be the perpendicular from the pole on the tangent, it is frequently convenient to express the equation to the curve in terms of r and p. Such an equation is, as will be subsequently seen, a *differential* one; but expressing as it does an essential property of a curve, it is sufficient to individualize it, and thus to be a mathematical definition. An equation for the involute of the circle can be easily obtained in this form.

Let $\text{SA} = a$, the radius of the circle; $\text{SP} = r$; $\text{SY} = p$; and let A be the point at which P, the generating point of the involute, is in contact with the circle.

Then from the geometry it is plain that QPY is a right angle, and that therefore QP is parallel to SY; whence we have

$$\text{SP}^2 = \text{SY}^2 + \text{PY}^2,$$
$$= \text{SY}^2 + \text{SA}^2;$$
$$\therefore \quad r^2 = p^2 + a^2. \tag{5}$$

218.] To find the equation to the circle in terms of p and r, any point being the pole; fig. 88.

Let CA, radius of circle, $= a$; SC $= c$, S being pole; SP $= r$; SY $= p$.

$$\therefore \quad \text{SC}^2 = \text{SP}^2 + \text{PC}^2 - 2.\text{SP}.\text{PC}.\cos\text{SPC},$$
$$= \text{SP}^2 + \text{PC}^2 - 2.\text{SP}.\text{PC}.\cos\text{YSP},$$
$$c^2 = r^2 + a^2 - 2ra\frac{p}{r};$$
$$\therefore \quad p = \frac{r^2 + a^2 - c^2}{2a}. \tag{6}$$

Hence, if the pole be on the circumference, say at B, $c = a$, and the equation is

$$p = \frac{r^2}{2a}. \tag{7}$$

219.] To find the equation to the epicycloid, in terms of r and p.

From Art. 177, equation (37),

$$x = (a+b)\cos\theta - b\cos\frac{a+b}{b}\theta,$$
$$y = (a+b)\sin\theta - b\sin\frac{a+b}{b}\theta;$$
$$\therefore \quad x^2 + y^2 = r^2 = (a+b)^2 + b^2 - 2b(a+b)\cos\frac{a}{b}\theta,$$
$$r^2 = a^2 + 2b(a+b)\left\{1 - \cos\frac{a}{b}\theta\right\}. \tag{8}$$

Also differentiating and reducing

$$ds^2 = 2(a+b)^2\left\{1 - \cos\frac{a}{b}\theta\right\}d\theta^2 = \frac{a+b}{b}(r^2 - a^2)\,d\theta^2,$$
$$y\,dx - x\,dy = (a+b)(a+2b)\left\{\cos\frac{a}{b}\theta - 1\right\}d\theta$$
$$= -\frac{a+2b}{2b}(r^2 - a^2)\,d\theta.$$

Also by equation (44), Art. 186,

$$pds = ydx - xdy;$$

$$\therefore \quad p^2 = \frac{(a+2b)^2}{4b(a+b)}(r^2 - a^2). \tag{9}$$

SECTION 2.—*On Tangents and Normals to Polar Curves.*

220.] Let $r = f(\theta)$ be the general type of the explicit equation to polar curves, and let us assume the figure drawn in fig. 89 to be the normal form of such curves; which figure the student is recommended to examine carefully, for the values of the lines in connexion with it will be deduced from the geometry of it.

Let S be the pole, SX prime radius, APQ the curve, PSX $= \theta$, SP $= r$. Let XSP be increased by a small angle QSP $= d\theta$, then SQ $= f(\theta + d\theta) = r + dr$. From centre S with radius SP $= r$ describe the small arc PR, subtending $d\theta$.

$$\therefore \quad \text{PR} = r\,d\theta, \tag{10}$$

$$\text{RQ} = dr; \tag{11}$$

Let PQ, the element of the arc of the curve, be represented by ds;

$$\therefore \quad \text{PQ}^2 = \text{PR}^2 + \text{RQ}^2,$$

$$ds^2 = dr^2 + r^2\,d\theta^2. \tag{12}$$

Through the two points, P, Q, on the curve, let a straight line be drawn; then, when the two points become infinitesimally near to each other, the line becomes a tangent, in accordance with the definition of a tangent given in the last Chapter; and therefore, if Q and P are infinitesimally near, QPT is a tangent; through P draw the normal PG, and through S draw TSG perpendicular to the radius vector SP, and SY perpendicular to the tangent PT. The lengths PT and PG are respectively called the Polar Tangent and Polar Normal; SG is called the Polar Subnormal; ST the Polar Subtangent; and SY, the perpendicular from the pole on the tangent, is symbolized by p. These lines we proceed to determine.

Since $\tan \text{PQR} = \dfrac{\text{PR}}{\text{RQ}}$, we have, from (10) and (11),

$$\tan \text{PQR} = \frac{r\,d\theta}{dr}. \tag{13}$$

And since $\text{SPT} = \text{SQT} + \text{PSQ} = \text{PQR} + d\theta$; therefore, SPT and PQR being in general finite angles, and $d\theta$ being an infinitesimal angle, we must neglect $d\theta$ in the above equation, and write

$$\text{SPT} = \text{PQR},$$

and SPT is the angle contained between the curve and the radius vector;

$$\therefore \quad \tan \text{SPT} = \frac{r\,d\theta}{dr}, \tag{14}$$

$$\frac{\sin \text{SPT}}{r\,d\theta} = \frac{\cos \text{SPT}}{dr} = \frac{1}{ds}, \tag{15}$$

by reason of Preliminary Theorem I and equation (12) above.

Hence also the following values result:

$$\left.\begin{aligned} \text{ST} &= \text{Polar Subtangent} = \text{SP} \tan \text{SPT} = \frac{r^2 d\theta}{dr}, \\ \text{SG} &= \text{Polar Subnormal} = \text{SP} \tan \text{SPG} = \text{SP} \cot \text{SPT} = \frac{dr}{d\theta}, \end{aligned}\right\} \tag{16}$$

$$\left.\begin{aligned} \text{PT} &= \text{Polar Tangent} = \text{SP} \sec \text{SPT} = \frac{r\,ds}{dr}, \\ \text{PG} &= \text{Polar Normal} = \text{SP} \operatorname{cosec} \text{SPT} = \frac{ds}{d\theta}, \end{aligned}\right\} \tag{17}$$

$$\left.\begin{aligned} \text{SY} &= p = \text{SP} \sin \text{SPY} = \frac{r^2\,d\theta}{ds} = \frac{r^2 d\theta}{(dr^2 + r^2 d\theta^2)^{\frac{1}{2}}}, \\ \text{PY} &= \text{SP} \cos \text{SPY} = \frac{r\,dr}{ds} = (r^2 - p^2)^{\frac{1}{2}}. \end{aligned}\right\} \tag{18}$$

Similarly may the values of other lines be determined.

221.] The value of p may be put under another form which is often very convenient. Let u be the reciprocal of the radius vector, so that $u = \dfrac{1}{r}$, then

ve, $\dfrac{1}{p^2} = \dfrac{dr^2 + r^2 d\theta^2}{r^4 d\theta^2} = \dfrac{1}{r^2} + \dfrac{1}{r^4}\dfrac{dr^2}{d\theta^2}$;

substit ting, in terms of u,

$$\frac{1}{p^2} = u^2 + \frac{du^2}{d\theta^2}. \tag{19}$$

of p in (18) might also have been deduced as the expression for p in equation (44), Art. 186,

$$p = y\frac{dx}{ds} - x\frac{dy}{ds},$$

$$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\} \quad \therefore \quad \left.\begin{array}{l} dx = \quad \cos\theta - r\sin\theta\, d\theta \\ dy = dr\sin\theta + r\cos\theta\, d\theta \end{array}\right\}; \tag{20}$$

$$\therefore \quad y\,dx - x\,dy = -r^2 d\theta,$$

$$ds \quad \{dr^2 + r^2 d\theta^2\}^{\frac{1}{2}};$$

$$\therefore \quad p = \quad \frac{r^2 d\theta}{ds}. \tag{21}$$

222.] It is frequently necessary to express the geometrical quantities of Article 220 in terms of p and r.

By similar triangles PQR, SPY

$$\frac{RQ}{PY} = \frac{RP}{SY} = \frac{PQ}{SP};$$

$$\text{or} \quad \frac{dr}{\{r^2-p^2\}^{\frac{1}{2}}} = \frac{r\,d\theta}{p} = \frac{ds}{r}; \tag{22}$$

$$\therefore \quad ds = \frac{r\,dr}{(r^2-p^2)^{\frac{1}{2}}}, \tag{23}$$

$$r^2 d\theta = \frac{rp\,dr}{(r^2-p^2)^{\frac{1}{2}}}. \tag{24}$$

223.] Examples illustrative of the preceding theory.

Ex. 1. Spiral of Archimedes.

$$r = a\theta,$$

$$dr = a\,d\theta;$$

$$\therefore \quad \frac{dr}{a} = \frac{r\,d\theta}{r} = \frac{ds}{(a^2+r^2)^{\frac{1}{2}}},$$

$$\therefore \quad p = \frac{r^2}{(a^2+r^2)^{\frac{1}{2}}}, \qquad \text{polar normal} = (a^2+r^2)^{\frac{1}{2}},$$

$$\text{polar subnormal} = \frac{dr}{d\theta} = a.$$

Ex. 2. Circle, Pole being at extremity of a diameter.

$$r = 2a\cos\theta,$$

$$dr = -2a\sin\theta\,d\theta,$$

$$\frac{-dr}{2a\sin\theta} = \frac{r\,d\theta}{r} = \frac{ds}{2a};$$

$$\therefore \quad p = \frac{r^2}{2a} = r\cos\theta = \text{rectangular abscissa},$$

$$\frac{r\,d\theta}{dr} = -\frac{r}{2a\sin\theta} = -\cot\theta.$$

Ex. 3. Logarithmic Spiral; see fig. 86.

$$r = a^{\theta},$$

$$dr = \log_e a . a^{\theta} d\theta = \log_e a . r\,d\theta,$$

$$\frac{dr}{\log a} = \frac{r\,d\theta}{1} = \frac{ds}{\{1+(\log_e a)^2\}^{\frac{1}{2}}};$$

$$\therefore \quad \frac{r\,d\theta}{dr} = \frac{1}{\log_e a} = \tan \text{SPT},$$

which is a constant; and therefore the curve cuts all its radii vectores at a constant angle, and accordingly it is called the Equiangular Spiral.

Also $$p = \frac{r^2\,d\theta}{ds} = \frac{r}{\{(\log_e a)^2+1\}^{\frac{1}{2}}},$$

which may be written in the form

$$p = mr; \tag{25}$$

and this is the equation to the equiangular spiral in terms of p and r, and wherein m is the sine of the constant angle contained between the radius vector and the curve.

Ex. 4. The Lituus.

$$r = \frac{a}{\theta^{\frac{1}{2}}};$$

$$\therefore \quad dr = -\frac{a\,d\theta}{2\theta^{\frac{3}{2}}} = -\frac{r\,d\theta}{2\theta},$$

$$\therefore \quad \frac{dr}{1} = \frac{-r\,d\theta}{2\theta} = \frac{\pm\, ds}{(1+4\theta^2)^{\frac{1}{2}}},$$

$$\therefore \quad p = \frac{r^2\,d\theta}{ds} = \frac{2a^2 r}{(r^4+4a^4)^{\frac{1}{2}}}. \qquad (26)$$

Ex. 5. To find the relation between p and r in the Conic Sections, the focus being the Pole.

The general equation in terms of r and θ is

$$r = \frac{2ae}{1+e\cos\theta},$$

wherein $2a$ is the distance from the focus to the directrix. Taking formula (19),

$$u = \frac{1+e\cos\theta}{2ae},$$

$$\frac{du}{d\theta} = \frac{-\sin\theta}{2a};$$

$$\therefore \quad u^2 + \frac{du^2}{d\theta^2} = \frac{1}{4a^2e^2} + \frac{1}{4a^2} + \frac{\cos\theta}{2a^2e},$$

$$= \quad . \quad . \quad . \quad . \quad . \quad . \quad + \frac{1}{ae}\left(u - \frac{1}{2ae}\right);$$

$$\therefore \quad \frac{1}{p^2} = \frac{u}{ae} - \frac{1-e^2}{4a^2e^2},$$

$$\frac{1}{p^2} = \frac{1}{aer} - \frac{1-e^2}{4a^2e^2};$$

and the equation represents an ellipse, parabola, or hyperbola, according as e is less than, equal to, or greater than unity.

Hence the equation to the parabola becomes

$$p^2 = ar.$$

SECTION 3.—*On Asymptotes to Polar Curves.*

224.] On Rectilinear Asymptotes.

Curves referred to polar coordinates of course admit of rectilinear and curvilinear asymptotes, in the same manner as those referred to rectangular coordinates. As curvilinear asymptotes however are of little use in determining the course of a curve, we shall say nothing of them in general, but only describe one remarkable species, viz. the asymptotic circle.

As a rectilinear asymptote is a tangent to a curve at an infinite distance, the formulæ of Art. 220 enable us to determine it.

If for any *finite* value of θ, say $\theta = a$, r is infinite, then either the radius vector itself, or a line parallel to it, is an asymptote to the curve; and since the polar subtangent, which is equal to $r^2 \frac{d\theta}{dr}$, becomes in this case the perpendicular distance from the pole on the tangent, if the value of $\frac{r^2 d\theta}{dr}$, corresponding to $\theta = a$ and $r = \infty$, be finite, the line can be constructed; and if $\frac{r^2 d\theta}{dr} = 0$, the radius vector itself is the asymptote; but if it be equal to ∞, the asymptote, being at an infinite distance from the pole, cannot be constructed. An inspection of fig. 90 will render this plain; in which SP is the infinite radius vector, TL the asymptote, ST the value of $\frac{r^2 d\theta}{dr}$, when $\theta = a$, and $r = \infty$. If there are several values of θ for which r is infinite, there may be several rectilinear asymptotes. Hence, to determine them,

Find what finite values of θ render $r = \infty$. If the polar subtangent, corresponding to such infinite values of r and finite values of θ, be finite, then there are rectilinear asymptotes which may be constructed as explained above.

It is to be borne in mind that when $\frac{r^2 d\theta}{dr}$ is positive, the asymptote lies *below* the radius vector, as in fig. 90; and if it be negative, the asymptote lies *above* it, as in fig. 91.

Or in other words, according as $r^2 \frac{d\theta}{dr}$ is positive or negative, so is the perpendicular on the asymptote to be drawn *in consequentia* or *in antecedentia*.

225.] **Examples illustrative of the preceding theory.**

Ex. 1. To find the position of the Asymptotes of the Hyperbola, whose polar equation is

$$\frac{(\cos\theta)^2}{a^2} - \frac{(\sin\theta)^2}{b^2} = \frac{1}{r^2};$$

$$\therefore \quad r = \infty, \text{ when } \tan\theta = \pm\frac{b}{a}.$$

$\therefore$ The asymptotes are inclined to the prime radius at

$$\tan^{-1}\left(\pm\frac{b}{a}\right).$$

Also
$$r^2\frac{d\theta}{dr} = \pm\frac{ab\{b^2(\cos\theta)^2 - a^2(\sin\theta)^2\}^{\frac{1}{2}}}{(a^2+b^2)\sin\theta\cos\theta},$$

which is equal to 0, at the critical angles; both asymptotes accordingly pass through the pole.

Ex. 2. To determine the position of the Asymptote to the Conchoid of Nicomedes; see Art. 169, equation (18).

$$r = a\sec\theta + b;$$

$\therefore$ $r = \infty$, when $\theta = \frac{\pi}{2}$, the asymptote therefore is perpendicular to the prime radius.

Also since
$$\frac{1}{r} = \frac{\cos\theta}{a+b\cos\theta},$$

$$-\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{a\sin\theta}{(a+b\cos\theta)^2};$$

$$\therefore \quad \frac{r^2\,d\theta}{dr} = a, \qquad \text{when } \theta = \frac{\pi}{2}.$$

The asymptote therefore cuts the prime radius at right angles, and at a distance a from the pole in the positive direction; see fig. 36.

Ex. 3. To determine the Asymptotes to the Lituus.

$$r = \frac{a}{(\theta)^{\frac{1}{2}}};$$

$$\therefore \quad r = \infty, \text{ when } \theta = 0;$$

$$\frac{1}{r} = \frac{(\theta)^{\frac{1}{2}}}{a}, \quad \therefore \quad -\frac{1}{r^2}\frac{dr}{d\theta} = \frac{1}{2a(\theta)^{\frac{1}{2}}};$$

$$\therefore \quad \frac{r^2\, d\theta}{dr} = -2a\theta^{\frac{1}{2}} = 0, \text{ when } \theta = 0.$$

The prime radius therefore is an asymptote, as delineated in fig. 85.

Ex. 4. To determine the Asymptotes of the Reciprocal Spiral.

$$r = \frac{a}{\theta};$$

$$\therefore \quad r = \infty, \text{ when } \theta = 0.$$

Also

$$\frac{1}{r} = \frac{\theta}{a}, \quad \therefore \quad -\frac{1}{r^2}\frac{dr}{d\theta} = \frac{1}{a};$$

$$\therefore \quad \frac{r^2\, d\theta}{dr} = -a.$$

The asymptote therefore is a line parallel to the prime radius, at a distance a from it, to be measured in antecedentia, since $\frac{r^2\, d\theta}{dr}$ is negative; see fig. 84.

226.] On Asymptotic Circles.

Suppose that the equation to a polar curve is such that r approaches to a finite limit, say a, as θ is infinitely increased, then the curve approaches more and more nearly to a circle whose radius is a, which circle is said to be asymptotic to the curve; and if the curve approaches to it from the outside, the circle is called an interior asymptotic circle, and if from the inside, an exterior asymptotic circle.

Ex. 1. $$r = a\left(\frac{1+\theta}{\theta}\right),$$

which may be written under the form

$$r = a + \frac{a}{\theta}.$$

First, let θ be positive, then r is always greater than a; and when $\theta = 0$, r is ∞, and $r^2 \frac{d\theta}{dr} = -a$, shewing that the line parallel to the prime radius at a distance a above it is an asymptote to the curve; and when $\theta = \infty$, $r = a$; whence we have an interior asymptotic circle, such as is drawn in fig. 91.

Secondly, let θ be negative, then

$$r = a - \frac{a}{\theta},$$

and therefore when $\theta = 0$, $r = -\infty$, and r is negative as θ increases until $\theta = 1$, in which case $r = 0$, and thence r is always less than a until $\theta = \infty$, when $r = a$. Thus we have the curve dotted in the figure, and with an exterior asymptotic circle of radius SA $= a$; the continuity of the two branches of the curve is worth remarking.

Ex. 2. To determine the Asymptotic Circle to the Curve,

$$\theta(2ar - r^2)^{\frac{1}{2}} = 1;$$

$$\therefore \quad \theta = \infty, \text{ when } r = 2a, \text{ and when } r = 0.$$

Therefore the circle whose radius is $2a$ is an asymptote to the curve; and as r must be always less than $2a$, otherwise θ would be affected with $\sqrt{-}$, the asymptotic circle is exterior to the spiral.

SECTION 4.—*On Direction of Curvature, and Points of Inflexion.*

227.] On an inspection of the figures numbered 92 and 93 it is manifest that, if a curve referred to polar coordinates is concave towards the pole, as r increases, p increases also, and therefore $\frac{dr}{dp}$ is positive; and if the curve is convex towards the pole, as r increases, p decreases, and *vice versâ*, and therefore $\frac{dr}{dp}$ is negative. If therefore the equation to the curve is given in the form $r = f(\theta)$, in order to determine whether the curve is concave or convex towards the pole, we must transform the equation into one between r and p, by means of the relations

given in (19) or (21) of Art. 221, and thence find $\frac{dr}{dp}$; and for all values for which

$\frac{dr}{dp}$ is positive, the curve is concave towards the pole,

$\frac{dr}{dp}$ is negative, the curve is convex towards the pole;

and therefore if at any point $\frac{dr}{dp}$ changes sign by passing through 0 or ∞, at such a point the direction of curvature changes, and there is a point of inflexion; hence, to determine such points, equate $\frac{dr}{dp}$ to 0 and to ∞, and examine whether $\frac{dr}{dp}$ changes sign; if it does, there is a point of inflexion.

Ex. 1. To determine the point of Inflexion of the Lituus.

$$r = \frac{a}{\theta^{\frac{1}{2}}};$$

and by equation (26), Art. 223,

$$p = \frac{2a^2 r}{\{4a^4 + r^4\}^{\frac{1}{2}}};$$

$$\therefore \quad \frac{dp}{dr} = \frac{2a^2(4a^4 - r^4)}{(4a^4 + r^4)^{\frac{3}{2}}},$$

$= 0$, if $r = a\sqrt{2}$, and changes sign from $+$ to $-$; the curve therefore having been concave towards the pole for values of r less than $a\sqrt{2}$, changes its direction of curvature at that point, and becomes convex towards the pole; see fig. 85, SB $= a\sqrt{2}$.

Ex. 2. To prove that the Equiangular Spiral is always concave towards the Pole.

$$r = a^{\theta};$$

and by equation (25), Art. 223,

$$p = mr;$$

$$\therefore \quad \frac{dp}{dr} = m;$$

which is always positive, and therefore the curve is always concave towards the pole.

Section 5.—*On tracing Polar Curves by means of their Equations.*

228.] Having discussed all the peculiarities which curves referred to polar coordinates *generally* admit of, we are now in a condition to analyze the equations, and to give general rules for tracing the curves of which they are the mathematical expressions and definitions.

1) If the equation is of the form $r = f(\theta) \pm \phi(\theta)$, so that $r = f(\theta)$ is diametral to the curve to be traced, we had better trace separately the two curves $r = f(\theta)$ and $r = \phi(\theta)$, and then by addition and subtraction of the radii vectores trace the required curve. Thus suppose we have to draw the curve whose equation is $r = a(2 \pm \sin\theta)$, the circle whose radius is $2a$ is diametral to the required curve, and its radii are to be increased and diminished by $a \sin\theta$ corresponding to the several values of θ; see fig. 94.

2) Investigate the several values of θ which make $r = 0$, or $= \infty$; and in the latter case, if the value of θ be finite, determine whether the polar subtangent is finite or not, as this is the criterion whether the rectilinear asymptote can be constructed or not. Give such particular values to θ as the equation suggests, as e.g. if the equation involves a function of 3θ, put $\theta = 15°, 30°, 45°$, &c.; or if the equation involves a function of $\frac{\theta}{4}$, put $\theta = 60°, 90°, 120°, 180°$, and so on. In general give to θ such values that r may be constructed; and, by giving to θ the values 0 and $n\pi$, we find the values of r when the curve cuts the prime radius, or the prime radius produced backwards; and make r revolve in both directions.

3) It is convenient to find $\frac{dr}{d\theta}$, as it is the ratio of the corresponding increments of r and θ; and therefore, if it is positive, as θ increases, r increases; and, if it is negative, r decreases as θ increases, and *vice versâ*. And if $\frac{dr}{d\theta} = 0$, we have no increase of r corresponding to an increase of θ; that is, the curve is at right angles to the radius vector, which is also manifest from equation (14), Art. 220, because at such a point $\tan SPT = \infty$.

And if $\frac{dr}{d\theta} = 0$ and changes its sign, we have a maximum or minimum value of r, the point corresponding to which is called an *apse*; of which there are instances in the ellipse, if the focus be the pole, at the extremities of the major axis: and of the circle, if the centre be the pole, every point is an apse.

4) Nothing more need be said on the subject of rectilinear asymptotes and asymptotic circles; or

5) On the direction of curvature and points of inflexion.

229.] Hence then to trace a curve referred to polar coordinates,

I. Investigate, arrange, and tabulate with their proper signs, all the particular values of θ which render $r = 0$, and $= \infty$; or equal to a value that may be constructed without difficulty.

II. Find $\frac{dr}{d\theta}$; examine its sign, and the values of θ at which it is equal to 0, and to ∞, and whether it changes its sign; if it does, at such points there are maximum and minimum radii vectores.

III. Determine whether any finite values of θ render $r = \infty$; if so, find the value of $r^2 \frac{d\theta}{dr}$ corresponding to this value of θ, and construct the asymptote. Examine whether there is an asymptotic circle.

IV. Transform the equation into its equivalent between r and p; find $\frac{dr}{dp}$, and examine its sign, for the purpose of determining whether the curve is convex or concave towards the pole; also examine whether $\frac{dr}{dp}$ changes its sign by passing through 0 or ∞, for a point at which such a change takes place will be a point of inflexion.

V. Trace the curve in a similar manner, by making r to revolve in a negative direction.

230.] Examples illustrative of the preceding theory.

Ex. 1. Trace the Curve $r = a \sin 2\theta$.

$$\frac{dr}{d\theta} = 2a \cos 2\theta.$$

The curve has no rectilinear asymptotes, for no value of θ makes $r = \infty$; hence we may tabulate as follows:

	θ	r	$\frac{dr}{d\theta}$
1	0	$-, 0, +$	$+$
2	$\frac{\pi}{4}$	a	$+, 0, -$
3	$\frac{\pi}{2}$	$+, 0, -$	$-$
4	$\frac{3\pi}{4}$	$-a$	$-, 0, +$
5	π	$-, 0, +$	$+$

which values are sufficient to enable us to draw the curve.

We have thus examined the course of the curve through two right angles; and as $\sin 2\theta$ has passed through all its values, it is unnecessary to tabulate further, as the tracing point will describe equal curves in the other two quadrants.

Also the revolution of the radius vector in a negative direction produces the same curve. The curve is delineated in fig. 95.

It begins, as shewn by 1, from the pole s, and as θ increases r increases until $\theta = \frac{\pi}{4}$, where the radius vector attains to a maximum value a, as shewn by 2; afterwards the radius vector decreases, becomes equal to zero, when $\theta = \frac{\pi}{2}$, and passes into the fourth quadrant, because for all values of θ between $\frac{\pi}{2}$ and π r is negative, and, when $\theta = \frac{3\pi}{4}$, attains to a minimum value, viz. $-a$, and describing a loop exactly equal to that in the first quadrant, falls into the pole when $\theta = \pi$; afterwards in the third quadrant r is positive, so that the tracing point describes another equal loop in it, reaching a maximum when $\theta = \frac{5\pi}{4}$, and the curve falls into the pole when $\theta = \frac{3\pi}{2}$; after which the radius vector again becomes negative; and therefore, passing

through the fourth quadrant, describes the loop in the second quadrant, which is exactly equal to those which have been already traced out in the other three quadrants.

Ex. 2. Trace the Curve $r = a \sin 3\theta$.

$$\frac{dr}{d\theta} = 3a \cos 3\theta.$$

	θ	r	$\frac{dr}{d\theta}$
1	$\frac{\pi}{6}$	a	$+, 0, -$
2	$\frac{\pi}{3}$	$+, 0, -$	$-$
3	$\frac{\pi}{2}$	$-a$	$-, 0, +$
4	$\frac{2\pi}{3}$	$-, 0, +$	$+$
5	$\frac{5\pi}{6}$	$-a$	$+, 0, -$
6	π	$+, 0, -$	$-$

Whence the curve is manifestly that drawn in fig. 96. If the radius vector revolves in the negative direction, the same three loops will be traced out.

From this and the former examples it appears that of all curves whose equations are of the form

$$r = a \sin n\theta,$$

the curve consists of n loops if n is an odd number, and of $2n$ loops if n is an even number.

Ex. 3. Trace the Curve whose Equation is

$$r = a \sin \frac{\theta}{2},$$

$$\frac{dr}{d\theta} = \frac{a}{2} \cos \frac{\theta}{2};$$

therefore r is never greater than a, and $r = 0$, when $\theta = 0$, $= 2\pi$, $= \ldots\ldots = 2n\pi$.

Also $\frac{dr}{d\theta} = 0$, when $\theta = \pi$, $= 3\pi$, $= \ldots\ldots = (2n+1)\pi$.

y we have the following table:

	θ	r	$\frac{dr}{d\theta}$
1	0	$-, 0, +$	$+$
2	π	a	$+, 0, -$
3	2π	$+, 0, -$	$-$
4	3π	$-a$	$-, 0, +$
5	4π	$-, 0, +$	$+$

Hence the radius vector is zero, when $\theta = 0$, and attains a maximum value a, when $\theta = \pi$; whence it decreases, becoming 0, when $\theta = 2\pi$, until it reaches a minimum $-a$, when $\theta = 3\pi$; after which it increases, passing through zero, when $\theta = 4\pi$, and becomes a, when $\theta = 5\pi$; wherefore the curve is that drawn in fig. 97.

Ex. 4. Trace the Curve whose Equation is

$$r^2 = a^2 \{(\tan\theta)^2 - 1\};$$

$$\therefore \quad r = \pm a \{(\tan\theta)^2 - 1\}^{\frac{1}{2}};$$

therefore r cannot be constructed whenever $(\tan\theta)^2$ is less than 1. Also as r is affected with $\pm$, the pole is *the centre* of the curve.

And as $r = \infty$, when $\theta = \frac{\pi}{2}$ and $= \frac{3\pi}{2}$, we must find $r^2 \frac{d\theta}{dr}$ in order to determine the asymptote.

$$\therefore \quad \frac{1}{r} = \frac{\pm 1}{a\{(\tan\theta)^2 - 1\}^{\frac{1}{2}}},$$

$$-\frac{1}{r^2}\frac{dr}{d\theta} = \frac{\mp \tan\theta(\sec\theta)^2}{a\{(\tan\theta)^2 - 1\}^{\frac{3}{2}}};$$

$$\therefore \quad r^2\frac{d\theta}{dr} = \pm\frac{a\{(\tan\theta)^2 - 1\}^{\frac{3}{2}}}{\tan\theta(\sec\theta)^2} = \pm a, \text{ when } \theta = \frac{\pi}{2}, \text{ and} = \frac{3\pi}{2}.$$

Hence there are two asymptotes perpendicular to the prime radius, at distances $\pm a$ from the pole.

	θ	r	$\frac{dr}{d\theta}$
1	$\frac{\pi}{4}$	$\pm\sqrt{-}, 0, \pm$	$\pm$
2	$\frac{\pi}{2}$	$\pm, \infty, \pm$	$\pm$
3	$\frac{3\pi}{4}$	$\pm, 0, \pm\sqrt{-}$	$\pm$

The curve therefore is that delineated in fig. 98.

Ex. 5. Trace the Curve whose Equation is

$$r = a\frac{\theta^2}{\theta^2-1},$$

$$\frac{dr}{d\theta} = \frac{-2a\theta}{(\theta^2-1)^2}.$$

Also $r = \infty$, when $\theta = \pm 1$; therefore we must find $r^2\frac{d\theta}{dr}$ in order to determine the asymptote.

$$\frac{1}{r} = \frac{\theta^2-1}{a\theta^2},$$

$$-\frac{1}{r^2}\frac{dr}{d\theta} = \frac{2}{a\theta^3};$$

$\therefore\ r^2\frac{d\theta}{dr} = -\frac{a\theta^3}{2} = \mp\frac{a}{2}$, when $\theta = \pm 1$; the asymptotes therefore are inclined at ± 1 to the prime radius, and the perpendicular distances from the pole on them are $\mp\frac{a}{2}$.

Also, when $\theta = \infty$, $r = a$, and therefore the circle whose radius is a is asymptotic, and is an interior asymptotic circle, since r is greater than a.

As the radius vector revolves in the positive direction, $r = 0$, when $\theta = 0$, and is negative, and negatively increasing, until

1 $r = -\infty$, and changes to $+\infty$, approaching to ar asymptote, receding on one side of it, and re-
g on he other; after which, as θ increases, r decreases,
t att ns its least value a, when $\theta = \infty$.

Again, as r revolves in a negative direction, it must be mea-
ed ba ards from $\theta = 0$ to $\theta = -1$, at which latter angle
$= -\infty$, and then changes its sign to $+\infty$; that is, the
nches of the curve have approached the rectilinear asymp-
e, and cut it at infinity; and as θ increases, r continually
creases and approaches to the asymptotic circle, of which the
dius is a. See fig. 99, in which the dotted branches indicate
parts due to the *negative* revol tion of r.

Ex. 6. Trace the Curve w ose Equation is

$$r = a\,\frac{\theta + \sin\theta}{\theta - \sin\theta},$$

$$\frac{dr}{d\theta} = -\frac{2a(\theta\cos\theta - \sin\theta)}{(\theta - \sin\theta)^2};$$

if θ be positive, r is positive, since the arc is greater than its sine. And since for all values of θ in the first and second quadrants $\sin\theta$ is positive, and for values in the third and fourth quadrants $\sin\theta$ is negative, therefore in the first and second quadrants r is greater than a, and in the third and fourth r is less than a.

And since, when $\theta = 0$, $\sin\theta = \theta$,

$\therefore$ when $\theta = 0$, $r = \infty$; hence, to determine the corresponding polar subtangent,

$$r^2\frac{d\theta}{dr} = \frac{a}{2}\,\frac{(\theta + \sin\theta)^2}{\theta\cos\theta - \sin\theta} = \frac{0}{0}, \text{ when } \theta = 0,$$

$$= \frac{a}{2}\,\frac{2(\theta + \sin\theta)(1 + \cos\theta)}{-\theta\sin\theta} = \frac{0}{0}, \text{ when } \theta = 0,$$

$$= a\,\frac{(1 + \cos\theta)^2 - \sin\theta\,(\theta + \sin\theta)}{-\sin\theta - \theta\cos\theta} = \infty, \text{ when } \theta = 0;$$

$\therefore$ the rectilinear asymptote cannot be drawn.

When $\theta = \infty$, $r = a$, therefore there is an asymptotic circle whose radius is a. Hence we tabulate as follows:

	θ	r
1	0	$\frac{1}{0}$
2	$\frac{\pi}{2}$	$a\,\frac{\pi+2}{\pi-2}$
3	π	a
4	$\frac{3\pi}{2}$	$a\,\frac{3\pi-2}{3\pi+2}$
5	2π	a
6	$\frac{5\pi}{2}$	$a\,\frac{5\pi+2}{5\pi-2}$
7	3π	a
8	∞	a

It appears then that the curve starts from infinity, as delineated in fig. 100, and periodically, when $\theta = \pi, = 2\pi, = \ldots\ldots$ passes through the points A and B, which are the extremities of the diameter of the circle, whose centre is the pole and whose radius is a; to which circle the curve continually approaches, being outside in the first and second quadrants, and inside in the third and fourth. There is then this peculiarity, that the curve on the outside is gradually becoming nearer and nearer to the circle, and the curve on the inside is receding from the diameter as θ increases and approaching to coincidence with the circle.

CHAPTER XII.

ON CURVATURE OF PLANE CURVES.

SECTION 1.—*Curves referred to Rectangular Coordinates.*

231.] CONCEIVE a tangent to be drawn at a point on a plane curve, which is such that the curve lies entirely on one side of the tangent; then the curve is said to be *convex* towards that side of space on which the tangent lies, and *concave* towards the other side; such is our definition of concavity and convexity; and on such a conception were investigated in Chapter X, the analytical criteria for determining the direction of curvature. Let us moreover suppose that at the point of the curve under consideration there is no discontinuity, or indeterminateness of derived-functions, or point of inflexion; then, as the curve deviates from the tangent line, such a deviation may be greater or less, and the curve may be more or less bent; herein then we have a new affection, viz. amount of bending or of *curvature*, as it is called: the nature of which we propose to examine in the present Chapter.

And to consider it in another point of view; an infinitesimal element of the curve commencing from a given point being straight, it is in its length coincident with the tangent line at that point; and the next element being inclined at an angle to the former one, deviates from the tangent. Now let the two consecutive elements be of equal lengths, and from the extremity of the second let a perpendicular be drawn to the tangent: as this perpendicular is longer or shorter, so will the deviation be greater or less, and the curve be more or less bent.

These terms however are but relative; and accordingly it is necessary to fix on some standard with which to compare such amount of bending, and to investigate some means by which the comparison may be made.

The circle naturally suggests itself; whatever its curvature

or bending be, it is the same at all points of the same circle: this is evident on the principle of sufficient reason; and in different circles, as the radius varies so does the curvature. Now as the radius increases, the deviation from a straight line becomes less and less; and in the limit vanishes when the radius becomes infinite, (see Art. 159); and as the radius decreases, the curvature increases, and in the limit when the radius becomes zero, the curvature becomes infinite, for the circle becomes a point, and the curve at once returns into itself; the radius of the circle therefore, and its curvature or deviation from a straight line, are so related that one varies inversely as the other. If then r be the radius of a circle, the amount of deviation of that circle from a straight line is a function of the reciprocal of r; let us give some definite name to this deviation, and in order that we may have a measure of it, let us define it. *Curvature* is the name which we shall adopt, and our mathematical definition of it is the simplest function of the reciprocal of r, viz.:

$$\text{Curvature of a circle} = \frac{1}{r}.$$

Now when an arc of a circle is given, we can in many ways determine the radius of the circle; but when the arc is infinitesimal, the following is best adapted to our present conceptions.

The relation between an arc, the radius, and the angle subtended at the centre is

$$\text{arc} = \text{radius} \times \text{angle}.$$

If therefore x, y be the coordinates to any point on a circle, $x + dx$, $y + dy$, the coordinates to the point infinitesimally near to it, and ds be the arc between the two points; and if normals be drawn to the circle at the points (x, y), $(x + dx, y + dy)$, and $d\psi$ be the angle at the centre contained between these normals, and if r be the radius, then

$$ds = r\,d\psi, \tag{1}$$

$$r = \frac{ds}{d\psi}, \tag{2}$$

and the curvature of the circle $= \dfrac{d\psi}{ds}$.

232.] Let us now consider in what manner these principles may be extended to any plane curve.

Suppose that the two points (x, y), $(x + dx, y + dy)$ are on any plane curve, and are such that at neither is there a point of discontinuity or of inflexion; then, although the ratio given in (2) above may be no longer constant at all points of a curve, but may vary as we pass from one point to another, yet when the distance ds is infinitesimal, $\frac{d\psi}{ds}$ will assume some determinate value, which we may call the *curvature of the curve* at the point; perhaps it may be said that it is a measure of the *mean* curvature of the arc, but the difference between that and the *actual* curvature at the point (x, y) is infinitesimal, and therefore must be neglected, so that the two become identical.

Conceive then two normals to be drawn at two consecutive points of the curve, these will generally meet at a finite distance, and let $d\psi$ be the small angle included between them, and ρ be the distance from the curve of the point at which they intersect; then, by reason of (2), and introducing $\pm$ so as to make ρ in all cases positive, as it indicates an absolute length

$$\rho = \pm \frac{ds}{d\psi}; \tag{3}$$

From the analogy of the circle, ρ is called the *radius of curvature,* and the point at which the two consecutive normals intersect is called the *centre of curvature*; and therefore we have the following definition:

Def.—The distance from the curve at which two consecutive normals of a plane curve intersect, is called the radius of curvature of the curve at that point.

233.] To determine the analytical values of the radius of curvature.

Let the equation to the curve be $y = f(x)$, (see fig. 101), and P, Q be the two points on it at which the normals PΠ, QΠ are drawn, Π their point of intersection, which is therefore the centre of curvature; through G draw a line KG parallel to QΠ; then $\text{PQ} = ds$, $\text{P}\Pi\text{Q} = d\psi = \text{PGK} = d.\tan^{-1}\left(\frac{dx}{dy}\right)$, $\text{P}\Pi = \text{Q}\Pi$ $= \rho =$ radius of curvature;

$$\therefore \quad ds = \pm \rho d.\tan^{-1}\left(\frac{dx}{dy}\right), \tag{4}$$

$$= \pm \rho \frac{d^2x dy - d^2y dx}{dy^2} \frac{dy^2}{dx^2 + dy^2};$$

$$\therefore \quad ds^3 = \pm \rho\,(d^2x dy - d^2y dx), \tag{5}$$

$$\therefore \quad \rho = \frac{\pm ds^3}{d^2x dy - d^2y dx}; \tag{6}$$

which expression is the most general value of ρ, as in it neither x, y, nor s is an equicrescent variable.

If x be made equicrescent, $d^2x = 0$, and we have

$$\rho = \frac{\pm ds^3}{d^2y dx} = \pm \frac{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}; \tag{7}$$

which is the value of the radius of curvature most commonly used.

And, if y be an equicrescent variable, $d^2y = 0$, and

$$\rho = \frac{\pm ds^3}{d^2x dy} = \frac{\pm \left(1 + \frac{dx^2}{dy^2}\right)^{\frac{3}{2}}}{\frac{d^2x}{dy^2}}. \tag{8}$$

As to the $\pm$ signs with which the above values of ρ are affected, it must be observed that they entered originally in (3) so as to give a positive sign to ρ, and therefore that $\frac{ds}{d\psi}$ was to be affected with $+$ or $-$, according as s and ψ simultaneously increased and decreased, or as one increased when the other decreased. Now such simultaneous increase and decrease, as is plain from figs. 53 and 54, depends on the curvature of the curve being convex or concave downwards. In (7) therefore, considering the numerator to be positive, of curves which are concave downwards, $\frac{d^2y}{dx^2}$ is negative, and therefore for ρ to be positive in such a case, we must affect (7) with the negative sign; and then, if ρ is negative at last, it indicates that the radius of curvature is measured up from the curve.

234.] Examples illustrative of the preceding.

Ex. 1. To determine the Radius of Curvature of an Ellipse.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;$$

$$\therefore \quad \frac{dy}{dx} = -\frac{b^2 x}{a^2 y}, \qquad \therefore \quad 1 + \frac{dy^2}{dx^2} = \frac{a^4 y^2 + b^4 x^2}{a^4 y^2},$$

$$\frac{d^2y}{dx^2} = -\frac{b^4}{a^2 y^3};$$

$$\therefore \text{ by formula (7)} \quad \rho = -\frac{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}},$$

$$\rho = \frac{(a^4y^2 + b^4x^2)^{\frac{3}{2}}}{a^4 b^4};$$

$\therefore$ radius of curvature at extremity of axis major $= \dfrac{b^2}{a}$,

. . . . axis minor $= \dfrac{a^2}{b}$;

Cor.—Radius of curvature of circle $= a$.

Ex. 2. In all Curves of the second degree of the form

$$y^2 = 4mx + nx^2,$$

the radius of curvature varies as the cube of the normal.

$$y\frac{dy}{dx} = 2m + nx,$$

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = n,$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{-4m^2}{y^3}.$$

Now by equation (42), Art. 186, normal $= y\left\{1 + \dfrac{dy^2}{dx^2}\right\}^{\frac{1}{2}}$;

substituting which, we have

$$\frac{\text{radius of curvature}}{(\text{normal})^3} = -\frac{1}{y^3 \frac{d^2y}{dx^2}} = \frac{1}{4m^2}.$$

Q. E. D.

Ex. 3. To find Radius of Curvature of Cycloid, starting point being origin.

$$x = a \text{ versin}^{-1}\frac{y}{a} - \{2ay - y^2\}^{\frac{1}{2}},$$

$$\frac{dx}{dy} = \frac{y}{\{2ay - y^2\}^{\frac{1}{2}}}; \quad \therefore \quad 1 + \frac{dy^2}{dx^2} = \frac{2a}{y},$$

$$\frac{d^2y}{dx^2} = -\frac{a}{y^2},$$

$$\therefore \quad \rho = 2(2ay)^{\frac{1}{2}},$$

Therefore, and from Ex. 6, Art. 190, it appears that the radius of curvature is equal to twice the normal.

Ex. 4. In illustration of formula (6), let it be required to find the length of the Radius of Curvature of the Ellipse whose Equations are (see Art. 166),

$$x = a\cos\theta, \qquad y = b\sin\theta,$$

$$dx = -a\sin\theta\, d\theta, \qquad dy = b\cos\theta\, d\theta,$$

$$d^2x = -a\cos\theta\, d\theta^2, \qquad d^2y = -b\sin\theta\, d\theta^2;$$

$$\therefore \quad ds^2 = \{a^2(\sin\theta)^2 + b^2(\cos\theta)^2\}\, d\theta^2,$$

$$= \left\{\frac{a^2}{b^2}y^2 + \frac{b^2}{a^2}x^2\right\} d\theta^2,$$

$$d^2x\, dy - d^2y\, dx = -ab\, d\theta^3;$$

$$\therefore \quad \rho = \pm \frac{\{a^4y^2 + b^4x^2\}^{\frac{3}{2}}}{a^4b^4}.$$

235.] On comparing figures 47 and 101, it appears that

$$\tau + \psi = \frac{\pi}{2};$$

$$\therefore \quad d\tau + d\psi = 0,$$

$$\therefore \quad \rho = \pm\frac{ds}{d\psi} = \mp\frac{ds}{d\tau}; \tag{9}$$

$d\tau$, which is equal to TLT′, fig. 101, is the angle between two tangents drawn at consecutive points on the curve; it is therefore the angle at which two successive elements are inclined to each other, and is called *the angle of contingence*.

236.] We proceed to determine other values of ρ which are required in the sequel;

$$\because \quad ds^2 = dx^2 + dy^2; \tag{10}$$

$$\therefore \quad ds\, d^2s = dx\, d^2x + dy\, d^2y. \tag{11}$$

Also by (6),
$$\pm \frac{ds^3}{\rho} = dy\, d^2x - dx\, d^2y. \tag{12}$$

Squaring (11) and (12), and then adding, we have

$$ds^2 (d^2s)^2 + \frac{ds^6}{\rho^2} = (dx^2 + dy^2)\{(d^2x)^2 + (d^2y)^2\},$$

$$= ds^2\{(d^2x)^2 + (d^2y)^2\}; \tag{13}$$

$$\therefore \quad \frac{ds^4}{\rho^2} = (d^2x)^2 + (d^2y)^2 - (d^2s)^2, \tag{14}$$

$$\frac{1}{\rho^2} = \frac{1}{ds^4}\{(d^2x)^2 + (d^2y)^2 - (d^2s)^2\}. \tag{15}$$

Whence we have the following values for (ρ):

(a) Let s be equicrescent; then $d^2s = 0$,

$$\frac{1}{\rho^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2. \tag{16}$$

Also by virtue of (11), since $d^2y = -\dfrac{dx}{dy}\, d^2x$,

$$\frac{1}{\rho^2} = \left(\frac{d^2x}{ds\, dy}\right)^2 = \left(\frac{d^2x}{ds^2}\right)^2 \frac{ds^2}{dy^2}, \tag{17}$$

and
$$\frac{1}{\rho^2} = \left(\frac{d^2y}{ds\, dx}\right)^2 = \left(\frac{d^2y}{ds^2}\right)^2 \frac{ds^2}{dx^2}. \tag{18}$$

(β) Let x be equicrescent; then $d^2x = 0$, and therefore by reason of (11)

$$d^2s = \frac{dy}{ds}\, d^2y;$$

$$\therefore \quad \frac{1}{\rho^2} = \frac{dx^6}{ds^6}\left(\frac{d^2y}{dx^2}\right)^2,$$

introducing dx^4 into the denominator to shew that x is equicrescent.

(γ) Let y be equicrescent; then $d^2y = 0$, and by the same process as above

$$\frac{1}{\rho^2} = \frac{dy^6}{ds^6}\left(\frac{d^2x}{dy^2}\right)^2.$$

The last two values of ρ are the same as (7) and (8).

Also from (14), multiplying through by ds^2, and replacing $ds\, d^2s$ from (11), we have

$$\begin{aligned}\frac{ds^6}{\rho^2} &= (d^2x)^2 ds^2 + (d^2y)^2 ds^2 + (d^2s)^2 ds^2 - 2(d^2s)^2 ds^2,\\ &= (d^2x)^2 ds^2 + (d^2y)^2 ds^2 + (d^2s)^2 (dx^2 + dy^2)\\ &\qquad\qquad - 2\, ds\, d^2s\,(dx\, d^2x + dy\, d^2y),\\ &= (d^2x\, ds - d^2s\, dx)^2 + (d^2y\, ds - d^2s\, dy)^2,\\ &= ds^4 \left\{\left(d.\frac{dx}{ds}\right)^2 + \left(d.\frac{dy}{ds}\right)^2\right\};\end{aligned}$$

$$\therefore\quad \frac{ds^2}{\rho^2} = \left(d.\frac{dx}{ds}\right)^2 + \left(d.\frac{dy}{ds}\right)^2, \tag{19}$$

which is identical with (16) when s is equicrescent.

Hence also, and from (9), we have the following value for the angle of contingence:

$$d\tau = \pm\left\{\left(d.\frac{dx}{ds}\right)^2 + \left(d.\frac{dy}{ds}\right)^2\right\}^{\frac{1}{2}}. \tag{20}$$

237.] Also we have the following values for $\cos\psi$ and $\cos\tau$, which are the cosines of the angles between the direction of the radius of curvature and the coordinate axes.

From (11) and (12), eliminating d^2x, we have

$$ds\, dy\, d^2s \pm \frac{ds^3}{\rho}\, dx = d^2y\,(dy^2 + dx^2);$$

$$\therefore\quad \frac{ds^3}{\rho}.\, dx = ds^2\, d^2y - d^2s\, dy\, ds,$$

$$\frac{dx}{ds} = \frac{\rho}{ds}\, d\left(\frac{dy}{ds}\right) = \cos\tau. \tag{21}$$

Similarly, $$\frac{dy}{ds} = \frac{\rho}{ds}\, d\left(\frac{dx}{ds}\right) = \cos\psi. \tag{22}$$

, if s be equicrescent,

$$\cos\tau = \rho\,\frac{d^2y}{ds^2}, \tag{23}$$

$$\cos\psi = \rho\,\frac{d^2x}{ds^2}. \tag{24}$$

If the equation to the curve be given in the implicit

$$F(x, y) = c, \tag{25}$$

titute as follows in the general value of ρ given in

ng (25), we have

$$\left(\frac{dF}{dx}\right) dx + \left(\frac{dF}{dy}\right) dy = 0, \tag{26}$$

$$\left(\frac{d^2F}{dxdy}\right) dx\,dy + \left(\frac{d^2F}{dy^2}\right) dy^2 + \left(\frac{dF}{dx}\right) d^2x + \left(\frac{dF}{dy}\right) d^2y = 0; \tag{27}$$

$$\therefore\ \left(\frac{dF}{dx}\right) d^2x + \left(\frac{dF}{dy}\right) d^2y = -\left\{\left(\frac{d^2F}{dx^2}\right) dx^2 + 2\left(\frac{d^2F}{dxdy}\right) dx\,dy + \left(\frac{d^2F}{dy^2}\right) dy^2\right\}. \tag{28}$$

Now from (26),

$$\frac{dy}{\left(\frac{dF}{dx}\right)} = \frac{-dx}{\left(\frac{dF}{dy}\right)} = \frac{d^2x\,dy - d^2y\,dx}{\left(\frac{dF}{dx}\right) d^2x + \left(\frac{dF}{dy}\right) d^2y}, \tag{29}$$

the last following by reason of Preliminary Theorem I, operating upon the fractions successively by the factors d^2x and d^2y, and adding numerators and denominators; each of which fractions is again equal to (by reason of the same Preliminary Theorem)

$$\frac{\pm\,ds}{\left\{\left(\frac{dF}{dx}\right)^2 + \left(\frac{dF}{dy}\right)^2\right\}^{\frac{1}{2}}}. \tag{30}$$

Whence, equating (29) and (30), and replacing the numerator

and denominator of (29) by their equivalents from (28) and (5), we have

$$\frac{\pm\, ds}{\left\{\left(\frac{dF}{dx}\right)^2 + \left(\frac{dF}{dy}\right)^2\right\}^{\frac{1}{2}}} = \frac{1}{\rho}\,\frac{ds^3}{\left(\frac{d^2F}{dx^2}\right)dx^2 + 2\left(\frac{d^2F}{dx\,dy}\right)dx\,dy + \left(\frac{d^2F}{dy^2}\right)dy^2}$$

and, replacing dx, dy, ds in terms of their proportionals given in (29) and (30), we have finally

$$\frac{1}{\rho} = \pm\frac{\left(\frac{dF}{dy}\right)^2\left(\frac{d^2F}{dx^2}\right) - 2\left(\frac{dF}{dx}\right)\left(\frac{dF}{dy}\right)\left(\frac{d^2F}{dx\,dy}\right) + \left(\frac{dF}{dx}\right)^2\frac{d^2F}{dy^2}}{\left\{\left(\frac{dF}{dx}\right)^2 + \left(\frac{dF}{dy}\right)^2\right\}^{\frac{3}{2}}}. \quad (31)$$

For an example of this formula, take the equation to the hyperbola

$$F(x, y) = xy = k^2;$$

$$\left(\frac{dF}{dx}\right) = y, \qquad \left(\frac{dF}{dy}\right) = x,$$

$$\left(\frac{d^2F}{dx^2}\right) = 0, \qquad \left(\frac{d^2F}{dx\,dy}\right) = 1, \qquad \left(\frac{d^2F}{dy^2}\right) = 0;$$

$$\therefore\quad \frac{1}{\rho} = \pm\frac{2xy}{(x^2+y^2)^{\frac{3}{2}}}.$$

SECTION 2.—*On Evolutes of Plane Curves referred to Rectangular Coordinates.*

239.] We have thus far determined the length of the radius of curvature, and the cosines of the angles at which its direction is inclined to the coordinate axes; our object now is to determine the coordinates to the centre of curvature; and since it changes position as the point, at which the radius of curvature is drawn, moves continuously along the curve, it thereby describes a continuous curve, which it is our object now to determine, and which curve is, for a reason which will shortly appear, called its *evolute*.

Let the equation to the curve be $y = f(x)$, and x, y the coordinates to the point on it at which the radius of curvature

is drawn; ξ, η the coordinates to the centre of curvature, so that, in fig. 102, we have

$$\text{OM} = x, \qquad \text{ON} = \xi, \qquad \text{PTN} = \tau,$$
$$\text{MP} = y, \qquad \text{NH} = \eta, \qquad \text{PHR} = \psi,$$
$$\text{PH} = \rho;$$

then since $\sin\psi = \frac{dx}{ds}$, and $\cos\psi = \frac{dy}{ds}$, we have from the geometry of the figure

$$\left.\begin{aligned} \xi &= x + \rho\frac{dy}{ds}, \\ \eta &= y - \rho\frac{dx}{ds}, \end{aligned}\right\} \tag{32}$$

which formulæ determine the position of the centre of curvature corresponding to any point of the curve; and from eliminating x and y, between these equations and the equation to the curve, viz. $y = f(x)$, there will result an equation involving ξ and η, which will represent the locus of the centre of curvature.

The equations (32) assume various forms, according to the value given to ρ; i. e. whether we express ρ by one or other of its values (6), (7), (8), (16). Thus applying the value of the radius of curvature given in (7), when x is equicrescent, we have

$$\left.\begin{aligned} \xi &= x - \frac{1 + \frac{dy^2}{dx^2}}{\frac{d^2y}{dx^2}}\frac{dy}{dx}, \\ \eta &= y + \frac{1 + \frac{dy^2}{dx^2}}{\frac{d^2y}{dx^2}}. \end{aligned}\right\} \tag{33}$$

If the equation to the curve be given in the implicit form, the equations (32) must be modified according to the equation (31) and those by means of which (31) has been determined; but as the expressions are long, not often employed, and easily found, it is not worth while to enlarge our book by inserting them at length.

240.] Examples on Evolutes.

Ex. 1. To determine the Evolute to the Parabola whose Equation is

$$y^2 = 4ax,$$

$$\frac{dy}{dx} = \frac{2a}{y},$$

$$\frac{d^2y}{dx^2} = -\frac{4a^2}{y^3}.$$

Substituting which in (33),

$$\begin{aligned} \xi &= x + \frac{y^2+4a^2}{y^2}\,\frac{y^2}{4a^2}\,\frac{2a}{y}, \\ &= x + \frac{4ax+4a^2}{2a}, \\ &= 3x+2a; \end{aligned}$$

$$\therefore \quad x = \frac{\xi - 2a}{3},$$

$$\begin{aligned} \eta &= y - \frac{y^2+4a^2}{y^2}\,\frac{y^3}{4a^2}, \\ &= \frac{-y^3}{4a^2}; \quad \therefore \quad y = (-4a^2\eta)^{\frac{1}{3}}. \end{aligned}$$

Substituting which in the equation to the parabola, we have

$$\eta^2 = \frac{4}{27a}(\xi - 2a)^3.$$

The equation to a semi-cubical parabola, whose cusp is on the axis of x at a distance $2a$ from the vertex of the parabola; see fig. 103.

Ex. 2. To find the Equation to the Evolute of a Circle.

$$x^2 + y^2 = a^2,$$

$$\frac{dy}{dx} = -\frac{x}{y}; \quad \therefore \quad 1 + \frac{dy^2}{dx^2} = \frac{x^2+y^2}{y^2} = \frac{a^2}{y^2},$$

$$\frac{d^2y}{dx^2} = -\frac{a^2}{y^3};$$

$$\therefore \quad \left.\begin{aligned} \xi &= x - \frac{a^2}{y^2}\,\frac{y^3}{a^2}\,\frac{x}{y} = 0, \\ \eta &= y - \frac{a^2}{y^2}\,\frac{y^3}{a^2} = 0; \end{aligned}\right\}$$

$\therefore$ the centre itself is the evolute.

Ex. 3. To find the Equation to the Evolute of the Ellipse.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

$$\frac{dy}{dx} = -\frac{b^2 x}{a^2 y}; \quad \therefore \quad 1 + \frac{dy^2}{dx^2} = \frac{a^4 y^2 + b^4 x^2}{a^4 y^2},$$

$$\frac{d^2 y}{dx^2} = -\frac{b^4}{a^2 y^3};$$

$$\left.\begin{aligned} \therefore \quad \xi &= x - \frac{a^4 y^2 + b^4 x^2}{a^4 y^2} \, \frac{a^2 y^3}{b^4} \, \frac{b^2 x}{a^2 y}, \\ &= x - \frac{x}{a^4 b^2}(a^4 b^2 - a^2 b^2 x^2 + b^4 x^2), \\ &= \frac{a^2 - b^2}{a^4} x^3, \end{aligned}\right\} \therefore \quad \frac{x^2}{a^2} = \left(\frac{a\xi}{a^2 - b^2}\right)^{\frac{2}{3}},$$

$$\left.\begin{aligned} \eta &= y - \frac{a^4 y^2 + b^4 x^2}{a^4 y^2} \, \frac{a^2 y^3}{b^4}, \\ &= y - \frac{y}{a^2 b^4}(a^2 b^4 - a^2 b^2 y^2 + a^4 y^2), \\ &= -\frac{a^2 - b^2}{b^4} y^3; \end{aligned}\right\} \therefore \quad \frac{y^2}{b^2} = \left(\frac{b\eta}{a^2 - b^2}\right)^{\frac{2}{3}},$$

whence, by addition,

$$(a\xi)^{\frac{2}{3}} + (b\eta)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}};$$

the curve represented by which is delineated in fig. 104.

Ex. 4. To find the Equation to the Evolute of the Cycloid.

Let the starting point be the origin; then

$$x = a \operatorname{versin}^{-1} \frac{y}{a} - (2ay - y^2)^{\frac{1}{2}},$$

$$\frac{dx}{dy} = \frac{y}{(2ay - y^2)^{\frac{1}{2}}}; \quad \therefore \quad 1 + \frac{dy^2}{dx^2} = \frac{2a}{y},$$

$$\frac{d^2 y}{dx^2} = -\frac{a}{y^2};$$

$$\therefore \quad \eta = y - \frac{2a}{y} \, \frac{y^2}{a} = -y,$$

$$y = -\eta;$$

$$\xi = x + \frac{2a}{y}\frac{y^2}{a}\frac{(2ay-y^2)^{\frac{1}{2}}}{y},$$

$$= x + 2(2ay - y^2)^{\frac{1}{2}},$$

$$= x + 2(-2a\eta - \eta^2)^{\frac{1}{2}},$$

$$\therefore \quad x = \xi - 2(-2a\eta - \eta^2)^{\frac{1}{2}};$$

substituting which values in the equation to the cycloid, we have

$$\xi - 2(-2a\eta - \eta^2)^{\frac{1}{2}} = a \operatorname{versin}^{-1}\frac{-\eta}{a} - (-2ay - \eta^2)^{\frac{1}{2}},$$

$$\xi = a \operatorname{versin}^{-1}\frac{-\eta}{a} + (-2a\eta - \eta^2)^{\frac{1}{2}},$$

which is the equation to a cycloid, when highest point is origin, equal to the original cycloid: ξ being parallel to the base, η measured along the axis in a negative direction, as is manifest on a comparison of the last equation with (31) in Art. 174, and of figs. 50 and 105; which last represents the positions of the original cycloid and its evolute.

Since $\eta = -y$, NH = MP, and therefore PH = 2PG, or the radius of curvature is equal to twice the normal; see Ex. 8, Art. 234. Also the radius of curvature at B = BC = 2BA = $4a$; also it is manifest that the radius of curvature at O = 0.

Ex. 5. To determine the Equation to the Evolute of the Equitangential Curve.

The equation to the curve is, see equation (27), Art. 178,

$$x = a \log\left\{\frac{a + (a^2-y^2)^{\frac{1}{2}}}{y}\right\} - (a^2-y^2)^{\frac{1}{2}},$$

$$\frac{dy}{dx} = -\frac{y}{(a^2-y^2)^{\frac{1}{2}}}; \quad \therefore \quad 1 + \frac{dy^2}{dx^2} = \frac{a^2}{a^2-y^2},$$

$$\frac{d^2y}{dx^2} = \frac{a^2y}{(a^2-y^2)^2};$$

$$\therefore \quad \eta = y + \frac{a^2}{a^2-y^2}\frac{(a^2-y^2)^2}{a^2y},$$

$$= \frac{a^2}{y}; \quad \therefore \quad y = \frac{a^2}{\eta};$$

$$\xi = x + \frac{a^2}{a^2 - y^2}\,\frac{(a^2-y^2)^2}{a^2 y}\,\frac{y}{(a^2-y^2)^{\frac{1}{2}}},$$

$$= x + (a^2-y^2)^{\frac{1}{2}},$$

$$= x + \left(a^2 - \frac{a^4}{\eta^2}\right)^{\frac{1}{2}},$$

$$\therefore \quad x = \xi - \frac{a}{\eta}(\eta^2 - a^2)^{\frac{1}{2}};$$

whence, substituting in equation above,

$$\xi - \frac{a}{\eta}(\eta^2-a^2)^{\frac{1}{2}} = a \log\left\{\frac{\eta + (\eta^2-a^2)^{\frac{1}{2}}}{a}\right\} - \frac{a}{\eta}(\eta^2-a^2)^{\frac{1}{2}};$$

$$\therefore \quad \frac{\xi}{a} = \log\left\{\frac{\eta+(\eta^2-a^2)^{\frac{1}{2}}}{a}\right\};$$

$$\therefore \quad \frac{\eta+(\eta^2-a^2)^{\frac{1}{2}}}{a} = e^{\frac{\xi}{a}},$$

$$\frac{\eta-(\eta^2-a^2)^{\frac{1}{2}}}{a} = e^{-\frac{\xi}{a}};$$

$$\therefore \quad \eta = \frac{a}{2}\left\{e^{\frac{\xi}{a}} + e^{-\frac{\xi}{a}}\right\},$$

the equation to the catenary, as is plain on comparing it with equation (25), Art. 172.

The relative position of the two curves is delineated in fig. 106; OM $= x$, MP $= y$; ON $= \xi$, NΠ $= \eta$; and since $\xi = x + (a^2-y^2)^{\frac{1}{2}}$, and NP $= a$, it follows that the ordinate through N (the foot of the tangent) passes through Π, the centre of curvature. Also since PΠ is the radius of curvature and PG is the normal of the equitangential curve, PN $\times$ PG $=$ NP^2 $= a^2$; also PΠ^2 $= \eta^2 - a^2 = \frac{a^4}{y^2} - a^2 = \frac{a^2}{y^2}(a^2-y^2)$.

Ex. 6. To determine the Equation to the Evolute of the Hypocycloid,

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}},$$

$$\frac{dy}{dx} = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}; \quad \therefore \quad 1 + \frac{dy^2}{dx^2} = \frac{a^{\frac{2}{3}}}{x^{\frac{2}{3}}},$$

$$\frac{d^2y}{dx^2} = \frac{a^{\frac{2}{3}}}{3x^{\frac{4}{3}}\,y^{\frac{1}{3}}};$$

$$\therefore \quad \xi = x + \frac{a^{\frac{2}{3}}}{x^{\frac{2}{3}}}\,\frac{3x^{\frac{4}{3}}y^{\frac{1}{3}}}{a^{\frac{2}{3}}}\,\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}},$$

$$= x + 3x^{\frac{1}{3}}y^{\frac{2}{3}}.$$

Similarly, $$\eta = y + 3x^{\frac{2}{3}}y^{\frac{1}{3}};$$

$$\therefore \quad \xi + \eta = x + 3x^{\frac{2}{3}}y^{\frac{1}{3}} + 3x^{\frac{1}{3}}y^{\frac{2}{3}} + y,$$

$$= (x^{\frac{1}{3}} + y^{\frac{1}{3}})^3,$$

$$(\xi + \eta)^{\frac{1}{3}} = x^{\frac{1}{3}} + y^{\frac{1}{3}}.$$

Similarly, $$(\xi - \eta)^{\frac{1}{3}} = x^{\frac{1}{3}} - y^{\frac{1}{3}},$$

$$\therefore \quad (\xi + \eta)^{\frac{1}{3}} + (\xi - \eta)^{\frac{1}{3}} = 2x^{\frac{1}{3}},$$

$$(\xi + \eta)^{\frac{1}{3}} - (\xi - \eta)^{\frac{1}{3}} = 2y^{\frac{1}{3}};$$

$$\therefore \quad \{(\xi+\eta)^{\frac{1}{3}}+(\xi-\eta)^{\frac{1}{3}}\}^2+\{(\xi+\eta)^{\frac{1}{3}}-(\xi-\eta)^{\frac{1}{3}}\}^2 = 4\{x^{\frac{2}{3}}+y^{\frac{2}{3}}\},$$

$$= 4a^{\frac{2}{3}},$$

$$(\xi + \eta)^{\frac{2}{3}} + (\xi - \eta)^{\frac{2}{3}} = 2a^{\frac{2}{3}}.$$

Let the coordinate axes be turned through 45°, so that

$$\xi = \frac{\xi' - \eta'}{\sqrt{2}}, \quad \text{and} \quad \eta = \frac{\xi' + \eta'}{\sqrt{2}};$$

$$\therefore \quad \xi + \eta = \xi' \sqrt{2},$$

$$\xi - \eta = -\eta' \sqrt{2};$$

$$\therefore \quad \xi'^{\frac{2}{3}} + \eta'^{\frac{2}{3}} = (2a)^{\frac{2}{3}}.$$

Accordingly the evolute is another hypocycloid, the radius of whose base-circle is twice that of the original circle, and whose cusps are on lines bisecting the lines joining the cusps of the original curve; see fig. 107. Similarly may the evolute of this new hypocycloid be found, which will be another hypocycloid, the radius of whose base-circle will be $4a$.

Theoretically, the equations to the evolutes of all curves may be found by means of equations (33), but the difficulty of elimination is in all cases, save in two or three besides the above, so great as to be beyond the present powers of analysis.

241.] We proceed now to discuss general properties of the curve whose current coordinates are ξ and η.

From equations (32), we have

$$\left.\begin{aligned} \xi - x &= \rho \frac{dy}{ds}, \\ \eta - y &= -\rho \frac{dx}{ds}; \end{aligned}\right\} \tag{34}$$

whence, squaring and adding,

$$(\xi - x)^2 + (\eta - y)^2 = \rho^2. \tag{35}$$

Multiplying the former by dx, and the latter by dy, and adding

$$(\xi - x)\, dx + (\eta - y)\, dy = 0. \tag{36}$$

Again, multiplying the former by d^2x, and the latter by d^2y, and adding

$$\begin{aligned} (\xi - x)\, d^2x + (\eta - y)\, d^2y &= \frac{\rho}{ds}\,(d^2x\, dy - d^2y\, dx), \\ &= ds^2, \end{aligned} \tag{37}$$

since by (5), $\rho(d^2x\, dy - d^2y\, dx) = ds^3$.

Now the relation between (35), (36), and (37) is very remarkable; for (36) is the differential of (35), and (37) is the differential of (36), the differentiations being calculated on the supposition that x and y vary, while ξ, η, and ρ do not change. And what *geometrical* fact is hereby implied? The following: (35) is the equation to a *circle* of which ρ is the radius, and the coordinates to whose centre are ξ and η, and of which x and y are the current coordinates. Hence the radius and coordinates to the centre of this circle remain the same, when for x and y we have successively $x + dx$, $y + dy$, and $x + 2\,dx + d^2x$, $y + 2\,dy + d^2y$; this circle therefore passes through *three* consecutive points on the curve, and therefore there are three points, and (what is the same thing) two consecutive elements, common to the circle and the curve. The circle is for an obvious reason called *the circle of curvature*. This result is in accordance with the principle of Art. 232; for although nothing was said as to a circle having points common with the curve, yet since ξ and η refer to the point of meeting of two consecutive normals, and each normal implies a tangent passing through two points, there must be three consecutive points in

the curve for which ξ, η, and ρ do not vary. It is also to be observed, that (36) is the equation to the normal if ξ and η are its current coordinates, and therefore that the centre of curvature is on the normal.

242.] Again, since the new curve is the locus of the point of intersection of any two consecutive normals of the original one, if the new curve be continuous, each normal must pass through two points in the new curve which are infinitesimally near to one another. Hence, in the expressions (34), ξ, η, ρ, x, y may all vary simultaneously, and we have

$$d\xi = dx + d\rho \frac{dy}{ds} + \rho \frac{d^2y\, ds - d^2s\, dy}{ds^2};$$

and substituting for ρ and d^2s from (5) and (11), we have

$$\left.\begin{aligned} d\xi &= \frac{dy}{ds}\, d\rho, \\ \text{and similarly,} \qquad d\eta &= -\frac{dx}{ds}\, d\rho. \end{aligned}\right\} \tag{38}$$

Whence it appears that we may differentiate (34) on the supposition that ρ, ξ, and η vary independently of x and y; that is, the normal passing through (x, y) passes through (ξ, η) and $(\xi + d\xi, \eta + d\eta)$, though of course, as is plain from the figure 102, the length of ρ varies.

243.] Squaring and adding the two equations of (38) we have

$$d\xi^2 + d\eta^2 = d\rho^2. \tag{39}$$

Let σ be the length of the arc of the new curve, and $d\sigma$ an element of it, then

$$d\sigma^2 = d\xi^2 + d\eta^2;$$

$$\therefore \quad d\sigma^2 = d\rho^2,$$

$$d\sigma = \pm\, d\rho. \tag{40}$$

And taking the positive sign, in order to accommodate the analytical expression to the curve in fig. 102, where AB $= \sigma$, and the element of the arc at B $= d\sigma$, and therefore σ and ρ are increasing simultaneously, we have

$$d\sigma - d\rho = 0;$$

$$\therefore \quad \sigma - \rho = \text{a constant} = c:$$

and therefore the difference in length between the radius of curvature of the original curve, and the length of the arc of the new curve is the same, at whatever point of the old curve the radius of curvature is drawn. Imagine then (see fig. 102) a perfectly flexible and inextensible string to be fixed at a point (ξ, η) of the new curve, say at Π, and of length equal to the radius of curvature of the old curve which abuts there, say equal to PΠ; then, if the string be wrapped round the curve, say towards A, just so much will be taken off from the string by the wrapping that the remainder will be equal to the radius of the old curve, corresponding to the point in the new curve at which the wrapping ends; and therefore if an inextensible string be unwrapped from the new curve, as e.g. from AΠ, the length of which is exactly equal to the length of the new curve + AO, which is constant, and is the radius of curvature of OP at O, the extremity of it will generate the old curve, viz. OP. It is for this reason that the new curve is called the *evolute**, as being that from which the string is unwrapped, and the original curve is called the *involute* with respect to it.

244.] It is manifest that the lengths of all evolutes can be determined; that is, the lengths can be compared with straight lines, whence they are said to be *rectifiable*; for, from what has preceded, the length of the evolute is equal to the difference of the radii of curvature of the involute corresponding to its two extremities.

Of this we subjoin some examples:

Ex. 1. To find the length of the Evolute of the Parabola, in terms of the Coordinates of its extremities.

$$\text{OM} = x, \quad \text{MP} = y; \quad \text{ON} = \xi, \quad \text{N}\Pi = \eta; \quad \text{see fig. 103.}$$

Let the equation to the parabola be $y^2 = 4ax$.

Then by equation (7), $\rho = \dfrac{2(a+x)^{\frac{3}{2}}}{a^{\frac{1}{2}}}$; therefore by what has preceded,

* By French writers the Evolute is named *développée*, and the Involute *développante*.

Length of AH = rad. of curv. at P — rad. of curv. at O;

$$= \frac{2(a+x)^{\frac{3}{2}}}{a^{\frac{1}{2}}} - 2a,$$

$$= \frac{2}{a^{\frac{1}{2}}}\left(\frac{a+\xi}{3}\right)^{\frac{3}{2}} - 2a;$$

since by Ex. 1, Art. 240, $x = \dfrac{\xi - 2a}{3}$.

Ex. 2. To determine the length of the fourth part of the Evolute of the Ellipse; see fig. 104.

Length of EC = rad. of curv. at B′ — rad. of curv. at A,

$$= \frac{a^2}{b} - \frac{b^2}{a}, \quad \text{by Ex. 1, Art. 234,}$$

$$= \frac{a^3 - b^3}{ab};$$

$\therefore$ length of whole evolute $= 4\dfrac{a^3 - b^3}{ab}$.

Ex. 3. To determine length of arc OH of Cycloid; see fig. 105.

Length of OH = PH,

$$= 2(2ay)^{\frac{1}{2}}, \quad \text{by Ex. 3, Art. 234,}$$

$$= 2(-2a\eta)^{\frac{1}{2}}, \quad \text{by Ex. 4, Art. 240;}$$

$\therefore$ if $\eta = -2a$, length of OC $= 4a$ = rad. of curv. at B,

$\therefore$ whole length of cycloid $= 8a$.

Ex. 4. To establish a relation between σ, η, and ξ in the Catenary.

In fig. 106, let AH $= \sigma$;

$$\therefore \quad \text{AH}^2 = \text{HP}^2,$$

$$\therefore \quad \text{AH}^2 = \text{HN}^2 - \text{NP}^2,$$

$$\sigma^2 = \eta^2 - a^2,$$

$$= \frac{a^2}{4}\left\{e^{\frac{\xi}{a}} + e^{-\frac{\xi}{a}}\right\}^2 - a^2,$$

$$= \frac{a^2}{4}\left\{e^{\frac{\xi}{a}} - e^{-\frac{\xi}{a}}\right\}^2,$$

$$\therefore \quad \sigma = \frac{a}{2}\left\{e^{\frac{\xi}{a}} - e^{-\frac{\xi}{a}}\right\}.$$

Every plane curve manifestly has an evolute, and has only one; but it has an infinite number of involutes, because in the unwrapping of the string which has been wound round the curve, every point of the stretched string describes a curve which is the involute corresponding to that point.

245.] Again, multiplying the former of (38) by dx, and the latter by dy, and adding, we have

$$dx\,d\xi + dy\,d\eta = 0; \tag{41}$$

$$\therefore \quad \frac{d\eta}{d\xi} = -\frac{dx}{dy}.$$

And since $\frac{d\eta}{d\xi}$ is the tangent of the angle made with the axis of x by the tangent to the evolute, and $\frac{dy}{dx}$ is the tangent of the angle between the axis of x and the tangent to the involute, it follows that the tangent to the evolute is perpendicular to the tangent to the involute, or that the normal to the involute is a tangent to the evolute. This result might have been anticipated from what is said in Art. 241.

246.] The following geometrical considerations will enable us better to understand some of the results we have arrived at in the preceding Articles on curvature.

The equations (35), (36), (37), connecting x, y, ρ, ξ and η, shew that the centre of curvature is the centre of a circle passing through three consecutive points in the curve. These *three* points are necessary to render the circle definite. If it passes through only two points, its centre may be *any where* on the normal which is perpendicular to the tangent passing through the two points, and thus there may be an infinite number of circles satisfying the condition: but if the circle is to pass through *three* points, its centre must be on the normal perpendicular to the tangent passing through the second and third points, as well as on the normal corresponding to the first and second points; and as these two normals will intersect in *one* point, this point must be the centre of the circle, and the circle becomes definite; in other words, the two consecutive elements of the curve which are delineated in fig. 108, viz. PQ and QR, will form two sides of a triangle, and by joining PR

the triangle will be completed, and the circle described about this triangle will be a definite circle, and pass through the points P, Q, R, which are three consecutive points on the curve. We may on this conception determine the radius of curvature as follows:

The angle of contingence, or $d\tau$, is the angle between the two consecutive elements PQ and QR, whence by equation (9), Art. 235,

$$d\tau = \pm \frac{ds}{\rho}. \tag{42}$$

Complete the parallelogram, of which PQ, QR are two adjoining sides, and draw the diagonal SQ, and the other lines as in the fig. 108; then

$$\text{PQ} = ds, \quad \text{QR} = ds + d^2 s,$$

$$\text{PK} = dx, \quad \text{QT} = dx + d^2 x = \text{PE}, \quad \because \text{ SP is parallel to RQ},$$

$$\therefore \text{ EK} = -d^2 x,$$

$$\text{QK} = dy, \quad \text{RT} = dy + d^2 y = \text{SE}, \quad \because \text{ SP is parallel to RQ},$$

$$\therefore \text{ SY} = d^2 y.$$

Therefore d^2x and d^2y are the projections of SQ on the coordinate axes.

Now area of parallelogram PQRS,

$$\begin{aligned} &= \text{PQ} \times \text{QR} \times \sin \text{PQR}, \\ &= ds\,(ds + d^2 s) \sin d\tau, \\ &= \pm \frac{ds^3}{\rho}, \end{aligned} \tag{43}$$

omitting d^2s when added to ds, and replacing $\sin d\tau$ by $d\tau$ in terms of equation (42).

Again, area of parallelogram PQRS,

$$\begin{aligned} &= \text{area of PQUV}, \\ &= \text{PK} \times \text{UQ} = \text{PK} \times \text{RW} = \text{PK}\,(\text{RT} - \text{WT}), \\ &= dx \left\{ dy + d^2 y - (dx + d^2 x)\frac{dy}{dx} \right\}, \\ &= dx\, d^2 y - dy\, d^2 x. \end{aligned} \tag{44}$$

Hence equating (43) and (44), we have

$$\frac{1}{\rho} = \pm \frac{d^2 x\, dy - d^2 y\, dx}{ds^3}. \tag{45}$$

247.] And again, making s equicrescent, so that in fig. 109 $\text{PQ} = \text{QR} = ds$, $\text{OM} = x$, $\text{MP} = y$, $\text{MN} = dx$, $\text{NL} = dx + d^2x$, $\text{NQ} = y + dy$, $\text{LR} = y + 2dy + d^2y$; let us complete as before the parallelogram PQRS, which is in this case equilateral; and therefore the radius of curvature lies along the line HSQ, which is coincident in direction with the diagonal SQ. Join PR, which is, as is plain from the geometry, perpendicular to SQ; then, if ρ be the radius of curvature, by the property of the circle,

$$2\rho : \text{RQ} :: \text{RQ} : \text{VQ};$$

$$\therefore \quad \rho = \frac{\text{RQ}^2}{2\,\text{VQ}} = \frac{ds^2}{\text{SQ}}. \tag{46}$$

Draw SY, YQ, as in the figure; then it may be proved, as in the last Article, that

$$\text{SY} = d^2y \quad \text{and} \quad \text{YQ} = -d^2x;$$

$$\therefore \quad \text{SQ}^2 = (d^2y)^2 + (d^2x)^2,$$

$$\therefore \quad \frac{1}{\rho^2} = \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2x}{ds^2}\right)^2. \tag{47}$$

Hence also we have a geometrical proof of the values of $\cos\tau$ and $\cos\psi$ determined in Art. 237, equations (23) and (24). As SQ is coincident in direction with the radius of curvature,

$$\therefore \quad \cos\psi = \frac{\text{YQ}}{\text{SQ}} = \rho\,\frac{d^2x}{ds^2},$$

$$\cos\tau = \frac{\text{SY}}{\text{SQ}} = \rho\,\frac{d^2y}{ds^2}.$$

248.] An inspection of the value of the radius of curvature, (7) in Art. 233, shews that the radius of curvature becomes infinite, or the curvature vanishes, whenever $\frac{d^2y}{dx^2} = 0$; that is, the curve degenerates into a straight line. Such is the case at a point of inflexion, at which also, as the sign of $\frac{d^2y}{dx^2}$ changes, the direction in which the radius of curvature is to be drawn changes also, and the evolute passes off into an infinite branch asymptotic to the normal at the point of inflexion, and passes round the sphere of which the plane in which the curve lies is the superior limit, as in figs. 24 or 25, whereby the evolute has

no point of discontinuity. If $\frac{d^2y}{dx^2} = 0$, and does not change its sign, then the branches of the evolute are related to the normal asymptote in the manner indicated in fig. 26; and if at any point on the curve $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ are both infinite, then the value of the radius of curvature must be determined by the methods explained in Chapter V.

Section 3.—*On Curvature of Plane Curves referred to Polar Coordinates.*

249.] To determine the length of the Radius of Curvature.

Let $r = f(\theta)$ be the equation to the curve (see fig. 110); then by equation (9), Art. 235,

$$\rho = \pm \frac{ds}{d\tau};$$

but τ = angle PKX = PSX + SPK,

$$= \theta + \tan^{-1}\frac{r\,d\theta}{dr};$$

$$\therefore\ d\tau = d\theta + \frac{d\left(\frac{r\,d\theta}{dr}\right)}{1+\frac{r^2d\theta^2}{dr^2}},$$

$$= \frac{r^2d\theta^3 + r\,dr\,d^2\theta + 2\,dr^2d\theta - r\,d\theta\,d^2r}{dr^2 + r^2d\theta^2}.$$

And since from equation (12), Art. 220,

$$ds = (dr^2 + r^2d\theta^2)^{\frac{1}{2}},$$

$$\rho = \pm\frac{(dr^2+r^2d\theta^2)^{\frac{3}{2}}}{r^2d\theta^3 + r\,dr\,d^2\theta + 2\,dr^2d\theta - r\,d\theta\,d^2r}; \qquad (48)$$

whence, if θ be equicrescent, $d^2\theta = 0$, and we have

$$\rho = \pm\frac{(dr^2+r^2d\theta^2)^{\frac{3}{2}}}{r^2d\theta^3 + 2\,dr^2d\theta - r\,d\theta\,d^2r}; \qquad (49)$$

and if r be equicrescent, $d^2r = 0$, and we have

$$\rho = \pm\frac{(dr^2+r^2d\theta^2)^{\frac{3}{2}}}{r^2d\theta^3 + r\,dr\,d^2\theta + 2\,dr^2d\theta}. \qquad (50)$$

al value (48) may also be deduced from that of Art. 233, by substituting

$$x = r\cos\theta,$$

$$y = r\sin\theta,$$

d cal ng the successive differentials of x and y, as is e in , Art. 71.

As an example of the above formula (49) let us take the wing:

Ex. 1. To find of the Radius of Curvature e.

$= a(1$

$$\frac{dr}{d\theta} = a\sin\theta, \qquad = \quad \cos\theta;$$

$$\therefore\ \rho^2 = \frac{\quad a^2(1-\cos\theta)^2\}^3}{\{a^2(1-\cos \quad \theta)^2 - a^2\cos\theta(1-\cos\theta)\}^2},$$

$$= \frac{(2a^2-2a^2 c}{\{3a^2(1-\cos}$$

$$= \frac{8}{9}\,ar.$$

Ex. 2. To find the length of the Radius of Curvature of the Spiral of Archimedes.

$$r = a\theta,$$

$$\frac{dr}{d\theta} = a, \qquad \frac{d^2r}{d\theta^2} = 0;$$

$$\therefore\ \rho = \pm\frac{(a^2+r^2)^{\frac{3}{2}}}{2a^2+r^2}.$$

250.] The expression for the radius of curvature may also be put under a very elegant and simple form, as follows:

Let the equation to the curve be changed into its equivalent in terms of r and p, by means of either (19) or (21) of Art. 221, so that

$$r = f(p).$$

Then, since $\tau = \theta + \text{SP}y$,

$$= \theta + \sin^{-1}\frac{p}{r};$$

$$\therefore \quad d\tau = d\theta + \frac{rdp - pdr}{r(r^2-p^2)^{\frac{1}{2}}};$$

and since by equations (23) and (24), Art. 222,

$$d\theta = \frac{pdr}{r(r^2-p^2)^{\frac{1}{2}}},$$

$$\text{and} \qquad ds = \frac{rdr}{(r^2-p^2)^{\frac{1}{2}}};$$

$$\therefore \quad d\tau = \frac{dp}{(r^2-p^2)^{\frac{1}{2}}}, \tag{51}$$

$$\therefore \quad \rho = \pm\frac{ds}{d\tau} = \pm\frac{rdr}{dp}. \tag{52}$$

251.] Which expression may also thus be obtained (see fig. 110):

$$\left.\begin{array}{l} \text{SP} = r, \\ \text{S}y = p, \end{array}\right\} \quad \text{PO} = \rho, \text{ the radius of curvature,}$$

O being the point at which two consecutive normals intersect. Draw Sz from S perpendicular to OP, then Pz = Sy = p; therefore from the geometry,

$$\text{SO}^2 = \text{SP}^2 + \text{PO}^2 - 2\,\text{SP}.\text{PO}.\sin \text{SP}y,$$

$$= r^2 + \rho^2 - 2r\rho\frac{p}{r},$$

$$= r^2 + \rho^2 - 2p\rho. \tag{53}$$

Then, since O and ρ remain the same for the next consecutive point in the curve, and r and p vary, we may differentiate on these conditions, and we have

$$0 = 2rdr - 2\rho dp;$$

$$\therefore \quad \rho = \frac{rdr}{dp}. \tag{54}$$

The geometrical meaning of which expression is evident from fig. 112.

Let Py, Py' be two tangents drawn at consecutive points on

the curve, sy, sy' the corresponding perpendiculars from the pole, then yPy' is the angle of contingence, and therefore

$$ds = \rho \times \text{angle } yPy';$$

but the angle yPy' is subtended by the small arc $sy' = dp$, at the distance Py', which is equal to $(r^2-p^2)^{\frac{1}{2}}$, therefore the angle yPy' may be replaced by

$$\frac{dp}{(r^2-p^2)^{\frac{1}{2}}};$$

and replacing ds by its value cited above, there results

$$\rho = \frac{r\,dr}{dp}.$$

252.] A comparison of the results we have arrived at with what has been said in Art. 241, shews that O is the centre of a circle which passes through three consecutive points on the curve. Let this circle be drawn as in fig. 111; then PV, the part of the radius vector SP which is intercepted by the circle, is called the *chord of the circle of curvature*. Its value is thus determined:

$$\begin{aligned} \text{PV} &= 2\text{PU} = 2\rho\cos\text{OPS}, \\ &= 2\rho\sin\text{SP}y = 2\rho\frac{p}{r}, \\ &= 2p\frac{dr}{dp}. \end{aligned} \tag{55}$$

253.] Examples illustrative of the preceding Articles.

Ex. 1. To find the lengths of the Radius and of the Chord of Curvature of the Logarithmic Spiral, whose Equation between r and p is

$$p = mr, \qquad \text{see equation (25), Art. 223,}$$

$$\frac{dp}{dr} = m;$$

$$\therefore \quad \rho = \frac{r}{m}; \qquad \text{chord} = 2mr\frac{1}{m} = 2r,$$

$$\therefore \quad \text{PV} = 2\text{PS}.$$

Ex. 2. The Involute of the Circle.

$$p^2 = r^2 - a^2,$$

$$p\frac{dp}{dr} = r;$$

$$\therefore \quad \rho = p, \qquad \text{chord of curvature} = \frac{2p^2}{r}.$$

Ex. 3. The Lemniscata of Bernoulli.

The polar equation is

$$r^2 = a^2 \cos 2\theta;$$

$\therefore$ in accordance with the notation of Art. 221,

$$a^2 u^2 = \sec 2\theta;$$

$$\therefore \quad 2a^2 u \frac{du}{d\theta} = 2 \sec 2\theta \tan 2\theta,$$

$$= 2a^2 u^2 \tan 2\theta,$$

$$\frac{du}{d\theta} = u \tan 2\theta;$$

$$\therefore \quad u^2 + \frac{du^2}{d\theta^2} = u^2 \{1 + (\tan 2\theta)^2\},$$

$$= u^2 (\sec 2\theta)^2,$$

$$= a^4 u^6;$$

$$\therefore \quad \frac{1}{p^2} = \frac{a^4}{r^6},$$

$$p = \frac{r^3}{a^2};$$

$$\therefore \quad \frac{dp}{dr} = \frac{3r^2}{a^2},$$

$$\therefore \quad \rho = \frac{a^2}{3r}, \qquad \text{chord of curvature} = \frac{2a^2 p}{3r^2} = \frac{2}{3} r.$$

Section 4.—*On Evolutes of Curves referred to Polar Coordinates.*

254.] The following is the most convenient method of determining the equation to the Evolute of a polar Curve.

Let the original equation be transformed into its equivalent between r and p, so that

$$r = f(p); \tag{56}$$

then, bearing in mind what has been said in Section 2 of the present Chapter on the properties of evolutes, which are general and independent of the particular system of determining position,

the line PO in fig. 111, which is a normal to the involute, is a tangent to the evolute; therefore, referring the evolute to polar coordinates, r' and p', if we take S to be the pole of the evolute, we have

$$\left.\begin{aligned} SO &= r' \\ SZ &= p' \end{aligned}\right\} \text{ coordinates to the evolute,}$$

and the following equations:

$$\rho = r\frac{dr}{dp}, \tag{57}$$

and from the geometry,

$$p' = (r^2 - p^2)^{\frac{1}{2}}, \tag{58}$$

and from equation (53), Art. 251,

$$r'^2 = r^2 + \rho^2 - 2\rho p;$$

from which four equations we can (theoretically at least) eliminate r, p, and ρ, and obtain a final equation involving only r' and p', which will be the polar equation to the evolute.

The several properties of evolutes, and the relation they bear to their respective involutes, might be deduced from a discussion of them referred to polar coordinates; but as they have been fully explained in the former part of this Chapter, it is not necessary to repeat them.

Ex. 1. To find the Equation to the Evolute of the Logarithmic Spiral.

$$p = mr;$$

$$\therefore \quad \frac{dp}{dr} = m,$$

$$\rho = \frac{r}{m};$$

$$\therefore \quad p' = (1 - m^2)^{\frac{1}{2}} r,$$

$$r'^2 = r^2 + \frac{r^2}{m^2} - 2r^2,$$

$$r' = \frac{(1-m^2)^{\frac{1}{2}}}{m} r;$$

$$\therefore \quad p' = mr',$$

which represents another equal logarithmic spiral.

The same result also follows from Ex. 1, Art. 253; for since PV $= 2$ PS, the perpendicular to SP from S meets the normal in the centre of curvature, or according to the type-figure, G is the centre of curvature; therefore the angle SGP, which is equal to SPT, is constant; and therefore as PG is a tangent to the evolute, the locus of G is a logarithmic spiral, the angle between the radius-vector and tangent of which is equal to the corresponding angle of the original curve; and therefore the two curves are equal.

Ex. 2. To determine the Evolute of the Involute of the Circle.

$$r^2 = p^2 + a^2,$$

$$\frac{dr}{dp} = \frac{p}{r}; \qquad \therefore \quad \rho = p;$$

$$\therefore \quad p' = a,$$

$$r'^2 = r^2 + p^2 - 2p^2,$$

$$= r^2 - p^2,$$

$$= a^2;$$

$$\therefore \quad r' = p' = a.$$

Hence the evolute is a circle whose radius is a, as is manifest from the figure 87.

CHAPTER XIII.

ON CONTACT OF CURVES AND ON ENVELOPES.

SECTION 1.—*On the Theory of Contact of Plane Curves.*

255.] IN Art. 182 we defined a tangent line to be that which passes through two consecutive points on a curve, and therefore it follows that there are two points *common* to the tangent and to the curve; and in Art. 241, we shewed that there were three points common to the curve and to the circle of curvature. This property of curves having consecutive points in common, or, as it is called, having *contact*, it is our object to generalize, and with reference to it we define as follows:

DEF.—Curves which have *two* consecutive points in common are said to have contact of the *first* order: those which have three consecutive points in common are said to have contact of the *second* order; and similarly two curves have contact of the nth order, when they have $(n+1)$ consecutive points in common.

Thus ordinarily, a tangent line has contact of the first order with a curve; and the circle of curvature has contact of the second order.

Curves which possess these relative properties are also called *osculating curves*, and curves are said to *osculate* to each other.

Nothing is said as to curves having only *one* point in common, because such a condition implies no more than that they intersect, and does not enable us to determine the relative direction of the curves.

256.] Hence then it appears that if for two curves whose equations are

$$y = f(x), \qquad \eta = \phi(\xi), \tag{1}$$

we have the series of common points indicated in the following table, viz.:

$\left.\begin{matrix}(x, y)\\(\xi, \eta)\end{matrix}\right\}$, the two curves intersect;

$\left.\begin{matrix}(x,y), \ (x+dx, \ y+dy)\\(\xi, \eta), \ (\xi+d\xi, \ \eta+d\eta)\end{matrix}\right\}$, the two curves have contact of the first order;

$\left.\begin{matrix}(x,y), \ (x+dx, y+dy), \ (x+2dx+d^2x, \ y+2dy+d^2y)\\(\xi, \eta), \ (\xi+d\xi, \eta+d\eta), \ (\xi+2d\xi+d^2\xi, \ \eta+2d\eta+d^2\eta)\end{matrix}\right\}$, there is contact of the second order;

and similarly as to contact of the nth order, for which it is necessary that the successive differentials of the variables up to the nth should be equal in both curves.

These conditions become greatly simplified if we consider x and ξ to be equicrescent, and each to increase by the same augment, in which case the several differentials of x and ξ, after the first, vanish, and we have, if

$\left.\begin{matrix}\xi = x\\\eta = y\end{matrix}\right\}$, intersection of the curves;

$\left.\begin{matrix}\xi = x\\\eta = y\end{matrix}\right\}$ and $\frac{d\eta}{d\xi} = \frac{dy}{dx}$, contact of the first order;

$\left.\begin{matrix}\xi = x\\\eta = y\end{matrix}\right\}$, $\frac{d\eta}{d\xi} = \frac{dy}{dx}$, and $\frac{d^2\eta}{d\xi^2} = \frac{d^2y}{dx^2}$, contact of the second order;

and if, besides, all the several successive differential coefficients, up to the nth inclusively, are equal in both curves, there is contact of the nth order.

Hence, if two curves have contact of the first order, they have a point in common, and the same tangent at the point; and therefore the tangent has contact of the first order. And if two curves have contact of the second order, they have not only a common tangent at the common point, but the curvature is the same, and is turned in the same direction; that is, they have the same circle of curvature.

257.] Hereby then the criterion of the order of contact assumes a new form; it depends on the number of the successively-derived functions of the equations to the curves which are equal; hence we are led to the following mode of viewing the subject, from which many important properties may be deduced.

Let $$y = f(x), \quad \eta = \phi(\xi) \tag{2}$$
be the equations to the two curves, which have a common point, $x = \xi$, $y = \eta$; and let y' and η' be the ordinates corresponding to the abscissa $x + h$; then

$$y' = f(x) + \frac{h}{1} f'(x) + \frac{h^2}{1.2} f''(x) + \ldots + \frac{h^n}{1.2\ldots n} f^n(x + \theta h), \tag{3}$$

$$\eta' = \phi(x) + \frac{h}{1} \phi'(x) + \frac{h^2}{1.2} \phi''(x) + \ldots + \frac{h^n}{1.2\ldots n} \phi^n(x + \theta h)^*. \tag{4}$$

Therefore if the contact be of the first order,

$$f(x) = \phi(x), \quad f'(x) = \phi'(x), \quad \text{and}$$

$$y' - \eta' = \frac{h^2}{1.2} \{f''(x + \theta h) - \phi''(x + \theta h)\}; \tag{5}$$

that is, if h be infinitesimal, the difference between the ordinates corresponding to $x + h$ is an infinitesimal of the second order.

And no other line can pass between these two curves, unless it has with each of them a contact of at least the first order; for suppose it were possible that the ordinate y_1 corresponding to the abscissa $x + h$ of the curve $y = \text{F}(x)$ should be such that $\text{F}'(x)$ is not equal to $f'(x)$, then

$$y' - y_1 = \frac{h}{1} \{f'(x + \theta h) - \text{F}'(x + \theta h)\}: \tag{6}$$

which difference is obviously greater than that given in (5), if h is infinitesimal; and therefore the curve $y = \text{F}(x)$ does not come between the curves $y = f(x)$ and $y = \phi(x)$.

If the contact be of the second order, then, besides the former conditions, we have

$$f''(x) = \phi''(x),$$

and subtracting (4) from (3), we have

$$y' - \eta' = \frac{h^3}{1.2.3} \{f'''(x + \theta h) - \phi'''(x + \theta h)\}; \tag{7}$$

that is, the difference between the ordinates corresponding to the abscissa $x + h$ is an infinitesimal of the third order.

* θ is used as the general symbol of a proper fraction, and does not necessarily represent the same quantity in (3) and (4), or in the subsequent equations: this is manifest from the argument of Chapter IV.

And if there is another curve, $y = F(x)$, such that $F''(x)$ is not equal to $f''(x)$, although $F'(x) = f'(x)$; then, if y_1 be the ordinate of this third curve corresponding to the abscissa $x+h$,

$$y' - y_1 = \frac{h^2}{1.2}\{f''(x+\theta h) - F''(x+\theta h)\}; \qquad (8)$$

which difference, being an infinitesimal of the second order, is obviously greater than that given by equation (7); and therefore this third curve does not come between the first two curves. Similarly, if the contact between two curves be of the nth order, the difference of the ordinates corresponding to $x+h$, when h is infinitesimal, is an infinitesimal of the $(n+1)$th order. Hence we have the following theorems:

"Two curves which have contact of the nth order are infinitely nearer to one another than two curves which have contact of an order lower than the nth."

"A curve which has contact of the nth order cannot come between two curves which have contact of an order higher than the nth."

"Two curves which have contact of the nth order with a third curve, have contact of at least that order with each other."

258.] An inspection of the equations (5) and (7) above, and of other equations formed in a similar manner, and giving the difference between the ordinates y' and η', corresponding to the several orders of contact, leads to the following theorem:

"If two curves have contact of an *odd* order, they touch and do not intersect; and if the contact be of an *even* order, the curves touch and intersect."

For suppose the contact to be of the nth order,

$$y' - \eta' = \frac{h^{n+1}}{1.2.3\ldots(n+1)}\{f^{n+1}(x+\theta h) - \phi^{n+1}(x+\theta h)\}. \qquad (9)$$

Then, if n be *even*, $y' - \eta'$ changes its sign as h changes sign; and therefore $f(x-h)-\phi(x-h)$ and $f(x+h)-\phi(x+h)$ have different signs, and therefore the curves intersect at the point of contact. But if the contact be of an *odd* order, n is odd and $n+1$ is even, and $y'-\eta'$ does not change sign with h; that is, the curve which was nearer to the axis of x before contact is nearer to it afterwards, and the curves do not intersect.

259.] Suppose that there are two curves, of one of which the equation is given, and contains certain fixed constants so that its form and position are completely determined, but that the equation to the other involves several arbitrary constants, in the determination of which we may make the curve fulfil certain conditions; we are going to shew that these latter may be satisfied by making the curve have with the former curve a contact, the order of which depends on the number of undetermined constants.

Let

$$F(x, y) = 0 \tag{10}$$

be the equation to the former curve; and

$$f(\xi, \eta, c_1, c_2 \ldots\ldots c_n) = 0; \tag{11}$$

that to the latter, in which $c_1, c_2 \ldots\ldots c_n$ are n arbitrary constants, and to be determined. From the theory of algebraical elimination it is plain, that there must be n independent equations to determine the n unknown quantities; let then n equations be formed by making the curve pass through n given points, that is, by substituting successively the coordinates to the given points for the current coordinates to the curve in its equation; and let us suppose these n points to be on the curve (10), and (which is allowable) to be infinitesimally consecutive points; then making x and ξ equicrescent, and variables with the same augments, in the two curves, by the latter part of Art. 256, all the successively-derived functions up to the $(n-1)$th of the equations of the two curves must be equal; up to the $(n-1)$th, I say, for thereby will n consecutive points be common, and sufficient conditions will have been introduced for the determination of all the constants.

And if the latter curve have with the former a contact of an order lower than the $(n-1)$th, there will not be sufficient conditions to determine all the n constants, and therefore the form and position of the latter curve may vary, and there may be any number of curves satisfying the conditions.

260.] Suppose that the equation to the latter curve contains but two arbitrary constants, being of the form

$$f(\xi, \eta, c_1, c_2) = 0; \tag{12}$$

then, differentiating

$$\left(\frac{df}{d\xi}\right) d\xi + \left(\frac{df}{d\eta}\right) d\eta = 0; \tag{13}$$

by means of which equations, when x and y are substituted for ξ and η, in combination with $F(x, y) = 0$, we may determine c_1 and c_2, and find a curve which will have contact of the first order with the latter curve.

Ex. 1. To determine the values of the constants a and b, that the straight line whose equation is

$$\frac{\xi}{a} + \frac{\eta}{b} = 1 \tag{14}$$

may have contact of the first order with the curve

$$F(x, y) = 0. \tag{15}$$

Since (14) is to pass through a point on (15), we have

$$\frac{\xi - x}{a} + \frac{\eta - y}{b} = 0. \tag{16}$$

Also differentiating (14),

$$\frac{d\xi}{a} + \frac{d\eta}{b} = 0, \tag{17}$$

and from (15),

$$\left(\frac{dF}{dx}\right) dx + \left(\frac{dF}{dy}\right) dy = 0, \tag{18}$$

and since $d\eta$ and $d\xi$ are to be equal to dy and dx, we have from the last equations

$$a \left(\frac{dF}{dx}\right) = b \left(\frac{dF}{dy}\right);$$

and multiplying (16) by the terms of this equality, there results

$$(\xi - x) \left(\frac{dF}{dx}\right) + (\eta - y) \left(\frac{dF}{dy}\right) = 0,$$

which is the equation to the tangent to the curve at the point, and which has therefore contact of the first order.

Also since from (17) $\dfrac{d^2\eta}{d\xi^2} = 0$, if at any point on the curve $\dfrac{d^2y}{dx^2} = 0$, the tangent will have contact of the second order; hence at a point of inflexion such is the case. Similarly if the several derived-functions of the equation to a curve up to the nth inclusive vanish at a certain point, the tangent at the point will have contact of the nth order with the curve.

Hence it appears that a straight line can generally have contact of only the first order with a curve, but may have contact of any order if the differential coefficients of the equation to the curve vanish at the point.

261.] If the equation to the latter curve involve three con- s, being of the form

$$f(\xi, \eta, c_1, c_2, c_3) = 0,$$

en, if ξ be equicrescent, we have

$$\left(\frac{df}{d\xi}\right) d\xi + \left(\frac{df}{d\eta}\right) d\eta = 0,$$

$$\left(\frac{d^2f}{d\eta^2}\right) \frac{d\eta^2}{d\xi^2} + 2 \left(\frac{d^2f}{d\eta d\xi}\right) \frac{d\eta}{d\xi} + \left(\frac{d^2f}{d\xi^2}\right) + \left(\frac{df}{d\eta}\right) \frac{d^2\eta}{d\xi^2} = 0.$$

By means of which equations, wh en x and y have been substituted for ξ and η, in addition to $F(x, y) = 0$, may the constants be eliminated, and the curve be determined which has contact of the second order with the given curve.

Ex. 1. To determine the conditions of contact of a Circle with a given Curve.

For the sake of simplicity let the equation to the given curve be in the explicit form

$$y = f(x); \tag{19}$$

and let the equation to the circle be

$$(\xi - a)^2 + (\eta - \beta)^2 = \rho^2, \tag{20}$$

in which three arbitrary constants are involved; hence ordinarily the circle may have contact of the second order with a given curve. Accordingly, differentiating twice, and taking neither variable to be equicrescent, we have

$$(\xi - a)\, d\xi + (\eta - \beta)\, d\eta = 0, \tag{21}$$

$$(\xi - a)\, d^2\xi + (\eta - \beta)\, d^2\eta + d\xi^2 + d\eta^2 = 0; \tag{22}$$

whence, by elimination

$$\xi - a = \frac{-d\eta\,(d\xi^2 + d\eta^2)}{d\eta\, d^2\xi - d\xi\, d^2\eta}, \tag{23}$$

$$\eta - \beta = \frac{d\xi\,(d\xi^2 + d\eta^2)}{d\eta\, d^2\xi - d\xi\, d^2\eta}; \tag{24}$$

and therefore
$$\rho^2 = \frac{(d\xi^2 + d\eta^2)^3}{(d\eta\, d^2\xi - d\xi\, d^2\eta)^2}; \tag{25}$$

and replacing η, ξ, and their differentials by y, x, and their differentials, and putting $dx^2 + dy^2 = ds^2$, we have

$$a = x + \frac{ds^2\, dy}{dy\, d^2x - dx\, d^2y}, \tag{26}$$

$$\beta = y - \frac{ds^2\, dx}{dy\, d^2x - dx\, d^2y}, \tag{27}$$

$$\rho^2 = \frac{ds^6}{(dy\, d^2x - dx\, d^2y)^2}, \tag{28}$$

which are severally the coordinates to the centre, and the radius of the circle which has contact of the second order with the curve.

If x be taken as an equicrescent variable, and the above expressions be modified accordingly, it will be seen on comparing them with the expressions marked (7) and (33) in the last Chapter, that they are the same as those determined for the coordinates of the centre of, and for the radius of curvature; and therefore the osculating circle, or that circle which has contact of the second order with a curve, is identical with the circle of curvature; and therefore all the properties which were in the last Chapter proved to belong to the circle of curvature, are also equally true of the osculating circle.

Hence, and from Art. 258, it appears that a circle, which has contact of the second order with a curve at a given point, cuts the curve as well as touches it; that is, as at such a point the curvature continuously varies without a singular value, the osculating circle is wholly within the curve on the side towards which the curvature decreases, and is without it on that towards which the curvature increases.

262.] Let us consider whether under any, and if so, under what circumstances the osculating circle determined as above has contact of the third order with a given curve.

In (21) and (22), replacing ξ and η by x and y, we have

$$(x - a)\, dx + (y - \beta)\, dy = 0, \tag{29}$$

$$(x - a)\, d^2x + (y - \beta)\, d^2y + dy^2 + dx^2 = 0; \tag{30}$$

[illegible]fore, differentiating again, since the contact is to be of the [illegible] order,

$$(x-a)\, d^3x + (y-\beta)\, d^3y + 3\,(dx\, d^2x + dy\, d^2y) = 0, \quad (31)$$

hence, by cross-multiplication, from (29), (30) and (31), we have

$$[illegible]x\, d^2x + dy\, d^2y)\,(dx\, d^2y - dy\, d^2x) + (d^3x\, dy - d^3y\, dx)\,(dx^2 + dy^2) = 0; \quad (32)$$

since

$$ds^2 = dx^2 + dy^2,$$

$$\therefore \quad ds\, d^2s = dx\, d^2x + dy\, d^2y;$$

$\therefore$ (32) becomes

$$3\, d^2s\,(dx\, d^2y - dy\, d^2x) + ds [illegible] {}^3x\, dy - d^3y\, dx) = 0, \quad (33)$$

which condition must be sat[illegible], that the circle may have contact of the third order.

Also since

$$\rho = \frac{[illegible]}{dx \; [illegible] \; dy\, d^2x},$$

$$\therefore \quad d\rho = \frac{3\, ds^2\, d^2s\,(dx\, d^2 [illegible] - ds^3\,(dx\, d^3y - dy\, d^3x)}{[illegible] dy\, d^2x)^2}; \quad (34)$$

the numerator $= 0$, by virtue of equation (33); therefore $d\rho = 0$, and ρ is a maximum or a minimum, or a constant. Therefore if at any point of a curve the radius of curvature is a maximum or a minimum, or constant, the osculating circle at that point has contact of the third order at least.

And if at any point on a curve the curvature is a maximum or a minimum, so that the contact is of an odd order, the osculating circle touches but does not intersect the curve: falling entirely within it when the curvature is a maximum, and entirely without it when the curvature is a minimum. In general too, in a closed curve which has no points of inflexion or cusps, such as an ellipse, the curvature has at least one maximum and one minimum value, and necessarily has the same number of minima as of maxima.

263.] Hence also it appears that, as the complete equation of the second degree of two variables is

$$A\eta^2 + B\xi\eta + C\xi^2 + D\eta + E\xi + F = 0,$$

which involves apparently six constants, but of which only *five*

are arbitrary, as one fixes the standard of comparison, a central curve of the second degree, with all its terms complete, may have contact of the fourth order with a given curve; but as in the case of the parabola a relation is given amongst the constants, and therefore only four are arbitrary, a parabola can have contact of only the third order. Similarly with regard to the ellipse, if certain conditions are given, the number of disposable constants becomes diminished, and the order of contact is lowered; as for instance, an ellipse with its major axis parallel to the axis of x has for its equation

$$\frac{(\xi-a)^2}{a^2}+\frac{(\eta-\beta)^2}{b^2}=1,$$

with four arbitrary constants, a, β, a, b; and therefore there can be contact of only the third order. Similarly, if in addition the ellipse is to be of a given eccentricity, then a relation is given between a and b, and we have only three arbitrary constants, and there can be contact of only the second order.

Thus again suppose the problem to be, to determine the coordinates to the vertex and the latus rectum of a parabola, whose principal axis is parallel to the axis of x, and which has contact of the second order with the curve whose equation is

$$y=f(x).$$

Let a and β be the coordinates to the vertex of the parabola, and $4m$ the latus rectum, then

$$(\eta-\beta)^2=4m(\xi-a),$$

$$(\eta-\beta)\frac{d\eta}{d\xi}=2m,$$

$$(\eta-\beta)\frac{d^2\eta}{d\xi^2}+\frac{d\eta^2}{d\xi^2}=0;$$

whence, replacing ξ, η, and their differentials by x, y, and their differentials, we have

$$\beta=y+\frac{\frac{dy^2}{dx^2}}{\frac{d^2y}{dx^2}}, \tag{35}$$

$$4m = -\frac{2\left(\frac{dy}{dx}\right)^3}{\frac{d^2y}{dx^2}}, \tag{36}$$

$$a = x - \frac{\frac{dy}{dx}}{2\frac{d^2y}{dx^2}}. \tag{37}$$

[illegible] means of (35) and (37), in combination with the equation [illegible] the curve $y = f(x)$, we may find the locus of the vertex of [illegible] parabola, as the point of contact moves along the given curve.

Section 2.—*On the Theory of Envelopes.*

264.] In the discussion of the relative properties of evolutes and involutes, in Chapter XII, A[illegible]s. 245 and 242, it is proved that the normal to the in[illegible] is a tangent to the evolute; or in other words, that as [illegible]ormal to the involute passes through two consecutive points of the evolute, the latter curve may be conceived to be made up of an infinite number of infinitesimal straight lines, each of which is a part of a normal to the involute; thus we say that the evolute is formed by the intersection of consecutive normals. Similarly, if a system of straight lines infinite in number, and varying in position infinitesimally from each other, is drawn, so that the perpendicular from a given point on each of them is the same, then the curve formed by the intersection of all such is a circle. Or again, suppose that an infinite number of equal circles have their centres along a straight line, and infinitesimally near to each other, then they all intersect in, and by their intersections form, two straight lines parallel to the given line. Curves formed in this manner, by the ultimate intersection of straight lines or curves drawn according to some given law, are called *envelopes*, and are said to envelope the *family* of curves. The general theory of them we proceed to discuss.

It is plain from algebraical as well as geometrical reasoning, that if an equation to a curve be given, involving one or more constants, as well as the current coordinates, the position and

dimensions of the curve will be changed by a change in the constants, and yet the class may remain the same; that is, a variation of a constant may involve a specific though not a generic change of curve. A constant that enters into an equation, and varies in the way above explained, is called a *variable parameter*. Thus in the equation to a parabola, $y^2 = 4mx$, as m varies, the form of the parabola will vary, though its vertex and principal axis are unaltered. In the equation of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

a and b may be variable parameters; in which case, changes of them will involve a change of an individual ellipse, though the class or family represented by the equation will remain that of ellipses still.

265.] Let the equation to the family of curves, of which it is our object to determine the envelope, involve only one parameter, and be

$$\text{F}(x, y, a) = 0, \tag{38}$$

in which a is the variable parameter, so that for every value of a we have some particular curve; but so that, if we make a to vary infinitesimally and continuously, we shall have a series of curves, the position of each one differing infinitesimally from that of the next. Suppose that a receives a variation da, then the two curves whose equations are (38) and

$$\text{F}(x, y, a + da) = 0, \tag{39}$$

are in position infinitesimally near to another; but owing to the variation of a they will in general intersect in some point, which will be determined by x and y being the same in both (38) and (39), and which will be a point on the envelope. If therefore we eliminate a between (38) and (39), the resulting equation will involve only x and y, and will be the equation to the envelope.

Before however we proceed to apply the method, we may put (39) under a more convenient form. By equation (21), Art. 101,

$$\text{F}(x, y, a + da) = \text{F}(x, y, a) + da\,\text{F}'(x, y, a + \theta da) = 0,$$

and therefore by reason of equation (38),

$$\text{F}'(x, y, a + \theta da) = 0,$$

and therefore in the limit, when da is infinitesimal,

$$F'(x, y, a) = 0. \tag{40}$$

To determine therefore the envelope of the family of curves whose general equation is $F(x, y, a) = 0$, and of which the several individuals are formed by making a to vary, we must eliminate a between $F = 0$, and $\frac{dF}{da} = 0$.

The geometrical conception of such envelopes evidently requires that the particular curve and the envelope should have the same tangent at their common point. And this truth is also proved as follows:

Differentiate (38), making x, y and a to vary, then

$$\left(\frac{dF}{dx}\right) dx + \left(\frac{dF}{dy}\right) dy + \left(\frac{dF}{da}\right) da = 0; \tag{41}$$

but since by reason of (40) $\frac{dF}{da} = 0$, therefore, whether a varies or not, we have the same equation, viz.

$$\left(\frac{dF}{dx}\right) dx + \left(\frac{dF}{dy}\right) dy = 0,$$

whereby to determine $\frac{dy}{dx}$; and therefore the tangent is the same to the envelope and to each curve at their common point.

266.] Ex. 1. To determine the Equation to the Curve formed by the intersection of the straight lines whose Equation is

$$y = ax + \frac{m}{a},$$

where a varies.

Differentiate with respect to a, x and y being constant,

$$0 = x - \frac{m}{a^2};$$

$$\therefore \quad a = \pm \left(\frac{m}{x}\right)^{\frac{1}{2}},$$

$$\therefore \quad y = \pm \sqrt{(mx)} \pm \sqrt{(mx)},$$

$$= \pm 2 \sqrt{(mx)},$$

$$\therefore \quad y^2 = 4mx,$$

which is the equation to a parabola.

Ex. 2. To determine the Envelope of the series of straight lines, of which the general Equation is

$$y = ax + (a^2 a^2 + b^2)^{\frac{1}{2}},$$

wherein a varies.

Differentiate with respect to a,

$$0 = x + \frac{a^2 a}{(a^2 a^2 + b^2)^{\frac{1}{2}}},$$

$$\therefore \quad a = -\frac{b}{a}\frac{x}{(a^2 - x^2)^{\frac{1}{2}}};$$

substituting which in the general given equation, and reducing and squaring, we have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

On examination of these two examples it will be seen, that the determination of envelopes produced by straight lines is the inverse one to that of finding the equation to a tangent to a curve; for the two equations to the straight lines are those severally known by the name of the "magical" equations to the tangents of the parabola and the ellipse. In this case then we have the equation to the tangent given, and the problem is, to determine the curve; in the other case the equation to the curve is given, and we have to determine that to the tangent. Hence the method of envelopes has been sometimes called "the inverse method of tangents."

The geometrical property involved in Ex. 1 is, "From a point in the axis of x, at a distance m from the origin, lines are drawn cutting the axis of y; at the points of intersection other lines are drawn perpendicular to these; to find the envelope of these latter lines." And that involved in Ex. 2 is, "To find the envelope of a series of straight lines drawn, so that the product of the two ordinates at distances $\pm a$ from the origin may be equal to b^2."

Ex. 3. To determine the Envelope of all Parabolæ expressed by the Equation

$$y = ax - \frac{1 + a^2}{2p} x^2,$$

wherein a varies.

Differentiate with respect to a,

$$0 = x - \frac{a}{p} x^2;$$

$$\therefore \quad a = \frac{p}{x},$$

$$\therefore \quad x^2 = 2p\left(\frac{p}{2} - y\right);$$

the envelope therefore is another parabola, having its focus at the origin.

Ex. 4. It is required to find the Envelope of Normals drawn to a given Curve.

Let the equation to the curve be

$$y = f(x),$$

then the equation to the normal is

$$(\eta - y)\frac{dy}{dx} + \xi - x = 0. \qquad (42)$$

Differentiate, considering η and ξ, which are the current co-ordinates to the normal, to be constant,

$$-(\eta - y)\frac{d^2y}{dx^2} + 1 + \frac{dy^2}{dx^2} = 0. \qquad (43)$$

From which, and from (42), we have

$$\left.\begin{aligned} \eta &= y + \frac{1 + \dfrac{dy^2}{dx^2}}{\dfrac{d^2y}{dx^2}} \\ \xi &= x - \frac{1 + \dfrac{dy^2}{dx^2}}{\dfrac{d^2y}{dx^2}} \frac{dy}{dx} \end{aligned}\right\}, \qquad (44)$$

which are the same expressions as those of equations (33), Art. 239, and therefore the evolute is the envelope of the normals. The method pursued above is manifestly the same as that of Chapter XII, but expressed in different language.

267.] If the equation representing the family of curves involve many, say n, variable parameters, and these parameters are related by $(n-1)$ other and independent equations, which conditions are equivalent to there being only *one* variable parameter, instead of eliminating $(n-1)$ parameters, and then differentiating with respect to the remaining one, and proceeding as in the last Article, the following method is more elegant.

Let the equation to the family of curves be

$$\text{F}(x, y, a_1, a_2, a_3 \ldots\ldots a_n) = 0, \tag{45}$$

and let the $(n-1)$ equations of condition be

$$\left.\begin{aligned} f_1(a_1, a_2 \ldots\ldots a_n) &= 0, \\ f_2(a_1, a_2 \ldots\ldots a_n) &= 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad &\cdot \quad \cdot \quad \cdot \\ f_{(n-1)}(a_1, a_2 \ldots\ldots a_n) &= 0. \end{aligned}\right\} \tag{46}$$

Conceive $a_1, a_2, a_3 \ldots\ldots a_n$ to vary by infinitesimal variations, whereby we have

$$\left(\frac{d\text{F}}{da_1}\right) da_1 + \left(\frac{d\text{F}}{da_2}\right) da_2 + \ldots\ldots + \left(\frac{d\text{F}}{da_n}\right) da_n = 0, \tag{47}$$

$$\left.\begin{aligned} &\left(\frac{df_1}{da_1}\right) da_1 + \left(\frac{df_1}{da_2}\right) da_2 + \ldots\ldots + \left(\frac{df_1}{da_n}\right) da_n = 0, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &\left(\frac{df_{(n-1)}}{da_1}\right) da_1 + \left(\frac{df_{(n-1)}}{da_2}\right) da_2 + \ldots + \left(\frac{df_{(n-1)}}{da_n}\right) da_n = 0. \end{aligned}\right\} \tag{48}$$

Then, if x and y be the same in (45) and (47), they refer to the point in the envelope where the two particular curves of the family intersect; and therefore if the several variable parameters and their differentials can be eliminated, the resulting equation between x and y will represent the locus of the points of intersection, which will be the envelope required.

By equations (45) ...... (48) we have $2n$ different relations, from which $(2n-1)$ quantities, viz. $a_1, a_2, a_3 \ldots\ldots a_n$, $\frac{da_1}{da_n} \ldots\ldots \frac{da_{n-1}}{da_n}$, are to be eliminated; which is, of course, a possible problem. To solve it, multiply the $(n-1)$ equations in (48) by $(n-1)$ indeterminate multipliers $\lambda_1, \lambda_2, \lambda_3, \ldots\ldots \lambda_{n-1}$,

and s[illegible]nd (47) together; by which means there will be n conditions involved in one equation; and therefore we are at liberty to make n new conditions. Let these be that the coefficients of the differentials, viz. $da_1, da_2, \ldots\ldots da_n$, be equal to zero; whence we have

$$\left.\begin{aligned}
\left(\frac{d\mathrm{F}}{da_1}\right) + \lambda_1\left(\frac{df_1}{da_1}\right) + \lambda_2\left(\frac{df_2}{da_1}\right) + \ldots + \lambda_{n-1}\left(\frac{df_{n-1}}{da_1}\right) &= 0,\\
[illegible] + \lambda_1\left(\frac{df_1}{da_2}\right) + \lambda_2\left(\frac{df_2}{da_2}\right) + \ldots + \lambda_{n-1}\left(\frac{df_{n-1}}{da_2}\right) &= 0,\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots&\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots&\\
\left(\frac{d\mathrm{F}}{da_n}\right) + \lambda_1\left(\frac{df_1}{da_n}\right) + \lambda_2\left(\frac{df_2}{da_n}\right) + \ldots + \lambda_{n-1}\left(\frac{df_{n-1}}{da_n}\right) &= 0,
\end{aligned}\right\} \quad (49)$$

and between the equations (45), (46), and (49), which are $2n$ in number, we may eliminate the $(2n-1)$ quantities, $a_1, a_2 \ldots\ldots a_n, \lambda_1, \lambda_2 \ldots\ldots \lambda_{n-1}$, and ultimately arrive at an equation between x and y only, which will be the required envelope to the family of curves.

In the particular case in which the general equation contains only two variable parameters, and one equation is given connecting them, the result assumes a form identical with that considered in Art. 143; and therefore, as is therein shewn, the ratio of the coefficients of the above differentials in the two differential equations is constant.

268.] Ex. 1. To find the Envelope of a series of straight lines, such that the perpendicular from the origin on them is equal to a given straight line c.

Let a be the angle which the perpendicular on any one of the lines makes with the axis of x; then the equation to the line is

$$x \cos a + y \sin a = c; \qquad (50)$$

whence, differentiating with respect to a, and bearing in mind the condition

$$(\cos a)^2 + (\sin a)^2 = 1,$$

$$y \cos a - x \sin a = 0; \qquad (51)$$

whence, squaring and adding (50) and (51), we have

$$x^2 + y^2 = c^2,$$

the equation to a circle, which is manifestly correct.

Ex. 2. A straight line of given length slides down between two rectangular axes; to find the Envelope of the line in all positions.

Let c be the length of the line; a and b the intercepts of the axes of x and y by the line: then the equation to the line is

$$\frac{x}{a} + \frac{y}{b} = 1; \tag{52}$$

wherein a and b are connected by the equation

$$a^2 + b^2 = c^2. \tag{53}$$

Differentiating therefore (52) and (53) by making a and b to vary, we have

$$\frac{x}{a^2}\, da + \frac{y}{b^2}\, db = 0,$$

$$a\,da + b\,db = 0,$$

and therefore, by reason of the remark at the end of the last Article,

$$\frac{\frac{x}{a^2}}{a} = \frac{\frac{y}{b^2}}{b} = \frac{\frac{x}{a}}{a^2} = \frac{\frac{y}{b}}{b^2} = \frac{1}{c^2};$$

$$\therefore \quad a = x^{\frac{1}{3}}\, c^{\frac{2}{3}},$$

$$b = y^{\frac{1}{3}}\, c^{\frac{2}{3}};$$

$$\therefore \quad a^2 + b^2 = (x^{\frac{2}{3}} + y^{\frac{2}{3}})\, c^{\frac{4}{3}},$$

$$\therefore \quad x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}.$$

Which curve is drawn in fig. 45, and of which therefore the length of the tangent intercepted between the two rectangular axes is constant.

Ex. 3. To find the Envelope of a series of Concentric and Coaxal Ellipses, of which the area is constant.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

$$ab = c^2;$$

$$\therefore \quad \frac{x^2}{a^3}\, da + \frac{y^2}{b^3}\, db = 0,$$

$$\frac{da}{a} + \frac{db}{b} = 0;$$

$$\therefore \frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{1}{2},$$

$$\therefore \quad a = x\sqrt{2},$$

$$b = y\sqrt{2};$$

$$\therefore \quad xy = \frac{c^2}{2},$$

the equation to an hyperbola referred to its asymptotes as axes.

Ex. 4. To find the Envelope of a system of straight lines, whose Equation is

$$\frac{x}{a} + \frac{y}{b} = 1, \tag{54}$$

a and b being related by the equation

$$\frac{a}{l} + \frac{b}{m} = 1, \tag{55}$$

$$\frac{x}{a^2}\,da + \frac{y}{b^2}\,db = 0,$$

$$\frac{da}{l} + \frac{db}{m} = 0;$$

$$\therefore \quad \frac{\frac{x}{a^2}}{\frac{1}{l}} = \frac{\frac{y}{b^2}}{\frac{1}{m}} = \frac{\frac{x}{a}}{\frac{a}{l}} = \frac{\frac{y}{b}}{\frac{b}{m}} = 1,$$

$$\therefore \quad a = (lx)^{\frac{1}{2}},$$

$$b = (my)^{\frac{1}{2}},$$

$$\therefore \quad \left(\frac{x}{l}\right)^{\frac{1}{2}} + \left(\frac{y}{m}\right)^{\frac{1}{2}} = 1. \tag{56}$$

The equation to a parabola, referred to two tangents as coordinate axes, the intercepts of which by the curve are l and m.

The geometry of the problem is represented in fig. 112: OA $= a$, OB $= b$, OL $= l$, OM $= m$; therefore ML is the fixed line whose equation is (55), and of which a and b are current coordinates; and AB is the varying line whose equation is (54). The formation of the curve is manifest from the figure.

Ex. 5. A series of equal Circles have their Centres placed along a given straight line; it is required to find the Envelope.

Let the equation to the given straight line be

$$x \cos a + y \sin a = p,$$

and x, y being the coordinates to the centre of, and ξ, η the current coordinates to, the circles of radius c, their equation is

$$(\xi - x)^2 + (\eta - y)^2 = c^2;$$

whence, differentiating, we have

$$\cos a \, dx + \sin a \, dy = 0,$$

$$(\xi - x)\, dx + (\eta - y)\, dy = 0;$$

$$\therefore \quad \frac{\xi - x}{\cos a} = \frac{\eta - y}{\sin a} = \pm c = \frac{\xi \cos a + \eta \sin a - (x \cos a + y \sin a)}{(\cos a)^2 + (\sin a)^2}$$

$$= \xi \cos a + \eta \sin a - p;$$

$$\therefore \quad \xi \cos a + \eta \sin a = p \pm c,$$

the equation to two straight lines parallel to the given line, and at distances $\pm c$ from it.

Ex. 6. From a given point on the circumference of a Circle chords are drawn, and on these, as diameters, circles are described; it is required to find their Envelope.

In fig. 124 let S be the given point in the circumference of the circle: from which let the chord SQ be drawn; and on SQ, as a diameter, let the circle SPQ be described; it is required to find the envelope of all circles described similarly to SPQ.

Let $\text{SP} = r$, $\text{PSA} = \theta$, $\text{SA} = 2a$, $\text{QSA} = \theta'$;

$$\therefore \quad \text{SQ} = 2a \cos \theta',$$

and since

$$\text{SP} = \text{SQ} \cos \text{PSQ},$$

$$\therefore \quad r = 2a \cos \theta' \cos (\theta - \theta'); \qquad (57)$$

differentiating which with respect to θ', since SP remains the same when θ' varies, we have

$$0 = 2a \{\cos \theta' \sin (\theta - \theta') - \sin \theta' \cos (\theta - \theta')\};$$

$$\therefore \quad \sin (\theta - 2\theta') = 0,$$

whereby (57) becomes

$$r = 2a\left(\cos\frac{\theta}{2}\right)^2,$$

$$= a(1+\cos\theta),$$

which is the equation to the cardioid, and is the curve drawn in fig. 44.

Ex. 7. Again, suppose that on the radii vectores of the cardioid, as diameters, circles are described as in the last Example; and again, on the radii vectores of the envelope, as diameters, circles are described, and so on continually; it is required to prove that the Envelope ultimately is a circle whose radius $= 2a$.

Suppose in fig. 124 SQA to be a cardioid, and the circle SPQ to be described on SQ as a diameter; then, if SP $= r$, PSA $= \theta$, QSA $= \theta'$,

$$r = 2a\left(\cos\frac{\theta'}{2}\right)^2\cos(\theta-\theta').$$

Differentiating which with respect to θ',

$$0 = 2a\cos\frac{\theta'}{2}\left\{\sin(\theta-\theta')\cos\frac{\theta'}{2} - \cos(\theta-\theta')\sin\frac{\theta'}{2}\right\};$$

$$\therefore \quad 2a\cos\frac{\theta'}{2}\sin\left(\theta-\frac{3\theta'}{2}\right) = 0,$$

$$\therefore \quad \theta' = \frac{2\theta}{3};$$

and the equation to the envelope is

$$r = 2a\left(\cos\frac{\theta}{3}\right)^3.$$

And if a similar process be continued n times,

$$r = 2a\left(\cos\frac{\theta}{n}\right)^n;$$

and if $n = \infty$, $r = 2a$;

that is, the ultimate envelope is a circle whose centre is S and radius is $2a$, and which is dotted in the figure.

SECTION 3.—*On Caustics.*

269.] A particular class of envelopes formed by straight lines exists in optics which is of too great importance to be passed over in silence in the present Chapter; we will therefore first give some general notions of the formation of such envelopes which are called Caustics, and then consider some particular examples and general properties of them.

In fig. 118 suppose S to be a source of light from which rays proceed and fall on a highly polished surface, which is perpendicular to the plane of the paper, and of which AP is a section made by the paper; and let SP be a type-ray of such a system incident on the surface at P. Now it is a physical property of a ray that it is *reflected* or turned back in the direction PR, such that, if PO be the normal to the curve at P, the angle of incidence (as it is called) SPO is equal to the angle of reflexion RPO. The envelope of the lines of which PR is the type, is called the *caustic by reflexion* of the surface.

And again in fig. 122, suppose S to be the source of a system of rays, of which let SP be the type; and suppose the rays to fall on a medium different to that in which S is, of which the bounding surface is perpendicular to the plane of the paper: and of which let AP be the section made by the paper; then by a physical law of optics, called the *law of refraction,* the ray SP does not proceed in the same straight line, but at P is bent or *refracted* into the direction P*r*, which is so related to SP that, if *n*PN be the normal to the surface at P, $\sin \text{SPN} = \mu \sin r\text{P}n$, where μ is constant for the same media, but varies for different media; that is, the sine of the angle of incidence bears a constant ratio to the sine of the angle of refraction. The envelope of all the refracted rays is called the *caustic by refraction* of the given surface*.

270.] To determine the Caustic by Reflexion of a system of Parallel Rays falling on a plane curve.

Suppose the source of light to be at an infinite distance, such as the sun is, and therefore all the incident rays to be parallel; and first let us suppose them to be parallel to the axis of x.

* In the figure the line AP is straight, but the matter of the text is expressed as though it were a curve.

In fig. 114 let QP be the incident type-ray, and PR the reflected type-ray: PS being the reflecting curve, and PG its normal at the point P.

Let $y = f(x)$ be the equation to the reflecting curve, and η and ξ the current coordinates of the reflected ray; then the equation to PR is

$$\eta - y = \tan \text{PRG}\,(\xi - x),$$

of which straight line we have to find the envelope.

Since the angles of incidence and reflexion are equal, QPG = RPG, and therefore their complements are equal, viz. QPL = RPT; but QPL = PTG $= \tan^{-1}\frac{dy}{dx}$,

$$\begin{aligned} \therefore\quad \text{PRG} &= \text{RPT} + \text{RTP}, \\ &= 2.\text{RTP}, \\ &= 2\tan^{-1}\frac{dy}{dx}; \end{aligned}$$

$$\therefore\quad \tan \text{PRG} = \frac{2\dfrac{dy}{dx}}{1 - \dfrac{dy^2}{dx^2}}.$$

and therefore the equation to the reflected ray is

$$\eta - y = \frac{2\dfrac{dy}{dx}}{1 - \dfrac{dy^2}{dx^2}}(\xi - x); \tag{58}$$

$$\therefore\quad \xi - x = \frac{1}{2}\left\{\frac{dx}{dy} - \frac{dy}{dx}\right\}(\eta - y). \tag{59}$$

Differentiating with respect to x,

$$1 = \frac{1}{2}\left\{\frac{dx^2}{dy^2}\frac{d^2y}{dx^2} + \frac{d^2y}{dx^2}\right\}(\eta - y) + \frac{1}{2}\left\{\frac{dx}{dy} - \frac{dy}{dx}\right\}\frac{dy}{dx};$$

$$\therefore\quad \eta - y = \frac{\dfrac{dy^2}{dx^2}}{\dfrac{d^2y}{dx^2}}, \tag{60}$$

$$\xi - x = \frac{1}{2}\,\frac{\dfrac{dy}{dx}\left(1 - \dfrac{dy^2}{dx^2}\right)}{\dfrac{d^2y}{dx^2}}. \tag{61}$$

By means of which equations, and that to the reflecting curve, we may eliminate x and y, and thereby obtain a relation between ξ and η, which will be the equation to the caustic.

Ex. 1. Let the Reflecting Curve be the Parabola.

$$y^2 = 4mx,$$

$$\frac{dy}{dx} = \frac{2m}{y}, \qquad \frac{d^2y}{dx^2} = -\frac{4m^2}{y^3};$$

$$\therefore \quad \eta - y = -\frac{4m^2}{y^2}\frac{y^3}{4m^2} = -y,$$

$$\therefore \quad \eta = 0,$$

$$\xi - x = -\frac{1}{2}\frac{2m}{y}\frac{y^2-4m^2}{y^2}\frac{y^3}{4m^2},$$

$$= -\frac{4mx - 4m^2}{4m},$$

$$= -x + m,$$

$$\therefore \quad \xi = m;$$

all the reflected rays therefore pass through the focus, which is their envelope; the caustic therefore is a point.

Ex. 2. Let the Reflecting Curve be a Circle.

$$x^2 + y^2 = a^2,$$

$$\frac{dy}{dx} = -\frac{x}{y}, \qquad \frac{d^2y}{dx^2} = -\frac{a^2}{y^3};$$

$$\therefore \quad \eta - y = -\frac{x^2}{y^2}\frac{y^3}{a^2} = -\frac{x^2y}{a^2},$$

$$\therefore \quad \eta = y\left(1 - \frac{x^2}{a^2}\right) = \frac{y^3}{a^2},$$

$$\therefore \quad y = a^{\frac{2}{3}}\eta^{\frac{1}{3}},$$

$$\xi - x = \frac{1}{2}\frac{x}{y}\left(1 - \frac{x^2}{y^2}\right)\frac{y^3}{a^2};$$

$$\therefore \quad \xi = x\left\{\frac{a^2 + 2y^2}{2a^2}\right\} = x\,\frac{a^2 + 2a^{\frac{4}{3}}\eta^{\frac{2}{3}}}{2a^2},$$

$$\therefore \quad x = \frac{2a^{\frac{2}{3}}\xi}{a^{\frac{2}{3}} + 2\eta^{\frac{2}{3}}},$$

$$\therefore \quad \eta^{\frac{2}{3}} a^{\frac{4}{3}} + \frac{4a^{\frac{4}{3}}\xi^2}{(a^{\frac{2}{3}} + 2\eta^{\frac{2}{3}})^2} = a^2,$$

$$\therefore \quad \xi = \pm \frac{1}{2}(a^{\frac{2}{3}} + 2\eta^{\frac{2}{3}})\{a^{\frac{2}{3}} - \eta^{\frac{2}{3}}\}^{\frac{1}{2}};$$

the equation to an epicycloid, the radius of whose fixed circle is $\frac{a}{2}$, and of whose generating circle is $\frac{a}{4}$, as will appear on eliminating θ between the two equations (37), Art. 177, having first replaced b by $\frac{a}{4}$ and a by $\frac{a}{2}$. See fig. 119 and Art. 273, where the problem is solved in a more simple though less general manner.

The above examples are sufficient for illustration, but the difficulties of elimination are in most cases insurmountable; the semi-cubical parabola is however another curve admitting of solution.

271.] Again, suppose the incident rays to be parallel to the axis of y, (see fig. 115,) of which let MP be the incident type-ray, and PR the reflected type-ray; let $y = f(x)$ be the equation to the reflecting curve, and ξ, η be the current coordinates of the reflected ray; then the equation to PR is

$$\eta - y = \frac{1}{2}\left\{\frac{dy}{dx} - \frac{dx}{dy}\right\}(\xi - x),$$

of which line we have to determine the envelope.

Differentiating, and proceeding as in the last Article, it will be found that

$$\xi - x = -\frac{\frac{dy}{dx}}{\frac{d^2y}{dx^2}}, \tag{62}$$

$$\eta - y = \frac{1}{2}\,\frac{1 - \frac{dy^2}{dx^2}}{\frac{d^2y}{dx^2}}. \tag{63}$$

By means of which, and the equation to the curve, we may eliminate x and y, and determine a relation between ξ and η, which will be the equation to the caustic.

Ex. 1. $$y^2 = 4mx,$$

$$\frac{dy}{dx} = \frac{2m}{y},$$

$$\frac{d^2y}{dx^2} = -\frac{4m^2}{y^3},$$

$$\xi - x = \frac{2m}{y}\frac{y^3}{4m^2} = \frac{y^2}{2m} = 2x;$$

$$\therefore\ x = \frac{\xi}{3},$$

$$\eta - y = -\frac{1}{2}\frac{y^2-4m^2}{y^2}\frac{y^3}{4m^2} = \frac{m-x}{2m}y,$$

$$\therefore\ y = \frac{6m\eta}{9m-\xi},$$

$$\therefore\ \eta^2 = \frac{\xi}{27m}(9m-\xi)^2;$$

which represents a curve symmetrical with respect to the axis of x, passing through the origin where it touches the axis of y; with a double point on the axis of x at a distance $9m$ from the origin, and a loop between the origin and that point; and approaching to the semi-cubical parabola as an asymptotic curve.

Ex. 2. Find the Equation to the Caustic of the Cycloid, the incident rays being perpendicular to the base of the Curve.

Take the starting point o for the origin, as in fig. 117, then the equation to the curve is

$$x = a \operatorname{versin}^{-1}\frac{y}{a} - (2ay - y^2)^{\frac{1}{2}},$$

$$\therefore\ \frac{dy}{dx} = \frac{(2ay-y^2)^{\frac{1}{2}}}{y}, \qquad \frac{d^2y}{dx^2} = -\frac{a}{y^2},$$

$$\therefore\ \eta - y = -\frac{1}{2}\frac{2y^2-2ay}{y^2}\frac{y^2}{a},$$

$$= \frac{ay-y^2}{a};$$

$$\therefore \quad y = a - (a^2 - a\eta)^{\frac{1}{2}},$$

$$\xi - x = \frac{y}{a}(2ay - y^2)^{\frac{1}{2}},$$

$$\therefore \quad x = \xi - (a\eta)^{\frac{1}{2}} + (a\eta - \eta^2)^{\frac{1}{2}},$$

$$\therefore \quad \xi - (a\eta)^{\frac{1}{2}} + (a\eta - \eta^2)^{\frac{1}{2}} = a \operatorname{versin}^{-1}\left\{1 - \left(\frac{a-\eta}{a}\right)^{\frac{1}{2}}\right\} - (a\eta)^{\frac{1}{2}},$$

$$\xi + (a\eta - \eta^2)^{\frac{1}{2}} = \frac{a}{2} \operatorname{versin}^{-1} \frac{2\eta}{a},$$

$$\therefore \quad \xi = \frac{a}{2} \operatorname{versin}^{-1} \frac{2\eta}{a} - (a\eta - \eta^2)^{\frac{1}{2}};$$

which equation is that to a cycloid, of which the starting point is the origin, and the radius of whose generating circle is one-half of that of the generating circle of the original cycloid.

Ex. 3. The Logarithmic Curve.

$$y = e^x,$$

$$\frac{dy}{dx} = e^x, \qquad \frac{d^2y}{dx^2} = e^x;$$

$$\therefore \quad \xi - x = -1, \qquad \therefore \quad x = \xi + 1,$$

$$\eta - y = \frac{1}{2}\{e^{-x} - e^x\},$$

$$\therefore \quad \eta = \frac{1}{2}\{e^x + e^{-x}\},$$

$$= \frac{1}{2}\{e^{\xi+1} + e^{-(\xi+1)}\}.$$

The equation to the catenary, the coordinates to the lowest point of which are, abscissa $= -1$, ordinate $= 1$.

272.] The following general properties of Caustics by Reflexion, formed by a system of Parallel Rays, deserve consideration.

(1) The distance from the incident point in the reflecting curve to the point of intersection of two consecutive reflected rays, is equal to one-fourth of the chord of the circle of curvature at the point of incidence which is parallel to the incident ray.

Let PQ, fig. 116, be an incident ray and PR be the reflected ray, P′ being the point where the next consecutive ray cuts it, and which is therefore a point in the caustic; let the circle drawn in the figure, and of which H is the centre and PH is the radius, be the circle of curvature at the point P; then PF and PE are the chords of the circle through P which are parallel to the axes of x and y respectively, and let L and K be the bisecting points of PF and of PE.

Now according to the notation of Art. 239 PL is equal to $\xi - x$, and PK to $y - \eta$ of that Article,

$$\left.\begin{aligned} \therefore \quad \mathrm{PL} &= -\frac{1+\dfrac{dy^2}{dx^2}}{\dfrac{d^2y}{dx^2}}\frac{dy}{dx} \\ \therefore \quad \mathrm{PK} &= -\frac{1+\dfrac{dy^2}{dx^2}}{\dfrac{d^2y}{dx^2}} \end{aligned}\right\} \qquad (64)$$

But from the formulæ (60) and (61), Art. 270,

$$\begin{aligned} (\mathrm{PP}')^2 &= (y-\eta)^2 + (x-\xi)^2, \\ &= \left\{\frac{1}{2}\,\frac{1+\dfrac{dy^2}{dx^2}}{\dfrac{d^2y}{dx^2}}\,\frac{dy}{dx}\right\}^2 = \left(\frac{\mathrm{PL}}{2}\right)^2; \end{aligned}$$

$$\therefore \quad \mathrm{PP}' = \frac{1}{2}\,\mathrm{PL}.$$

Similarly, if the incident rays be parallel to the axis of y, and Q′ be the point of the caustic on the reflected ray, it may be shewn by means of equations (62) and (63), that

$$\mathrm{PQ}' = \frac{1}{2}\,\mathrm{PK}.$$

The expressions throughout would have assumed a more symmetrical though less simple form if we had not considered x to be equicrescent.

(2) If the radiating point be such that a normal to the reflecting curve can be drawn through it, the caustic corresponding to the point where the normal meets the curve ultimately becomes a semi-cubical parabola.

For if the part of the reflecting curve which receives the rays parallel or nearly parallel to the normal through the source of light be taken very small compared to the distance of the origin of light from the curve, the system may be supposed to be one of parallel rays; and also whatever the reflecting curve be, we may consider it to be identical with its circle of curvature at the point, so that the problem ultimately becomes, for the small distance, that solved in Ex. 2, Art. 270, wherein we may consider η to be small compared with a.

$$\therefore \quad 2\xi = (a^{\frac{2}{3}} + 2\eta^{\frac{2}{3}}) \{a^{\frac{2}{3}} - \eta^{\frac{2}{3}}\}^{\frac{1}{2}},$$

$$= \ldots\ldots a^{\frac{1}{3}} \left\{1 - \frac{\eta^{\frac{2}{3}}}{a^{\frac{2}{3}}}\right\}^{\frac{1}{2}},$$

$$= \ldots\ldots a^{\frac{1}{3}} \left\{1 - \frac{\eta^{\frac{2}{3}}}{2a^{\frac{2}{3}}} - \ldots\ldots\right\},$$

$$= a + \frac{3}{2} a^{\frac{1}{3}} \eta^{\frac{2}{3}} + \ldots\ldots$$

and neglecting terms involving powers of η higher than those retained, we have

$$2\xi - a = \frac{3}{2} a^{\frac{1}{3}} \eta^{\frac{2}{3}}.$$

The equation to a semi-cubical parabola, the vertex of which is at a distance $\frac{a}{2}$ from the origin. An examination of figs. 117 ...... 120 at the point C, renders plain the geometrical form of the problem.

273.] The general form of the Equation to a Caustic of a Circle by Reflexion may be most conveniently determined as follows:

In fig. 118 let S be the source of light, and SP the incident type-ray, and PR the reflected type-ray, O being the centre of the circle. Let OA $= a$, OS $= b$, SPO $=$ RPO $= \phi$, POA $= \theta$; then taking O as the origin, and OS as the axis of x, the equation to the reflected ray PR is

$$x \sin(\theta + \phi) - y \cos(\theta + \phi) = a \sin\phi,$$

or
$$x(\sin\theta \cot\phi + \cos\theta) - y(\cos\theta \cot\phi - \sin\theta) = a; \quad (65)$$

and from the geometry of the figure

$$b \sin(\theta - \phi) = a \sin\phi,$$

$$\therefore \quad \cot\phi = \frac{a + b\cos\theta}{b\sin\theta}; \tag{66}$$

whence (65) becomes

$$x(a\sin\theta + b\sin 2\theta) - y(a\cos\theta + b\cos 2\theta) = ab\sin\theta. \tag{67}$$

Differentiating with respect to θ, we have

$$x(a\cos\theta + 2b\cos 2\theta) + y(a\sin\theta + 2b\sin 2\theta) = ab\cos\theta; \tag{68}$$

whence, eliminating from (67) and (68), we have

$$\left.\begin{aligned} x &= \frac{ab(2a + 3b\cos\theta - b\cos 3\theta)}{2(a^2 + 3ab\cos\theta + 2b^2)} \\ y &= \frac{ab^2(3\sin\theta - \sin 3\theta)}{2(a^2 + 3ab\cos\theta + 2b^2)} \end{aligned}\right\}; \tag{69}$$

which are the equations to the caustic in terms of a subsidiary angle θ. In two cases they reduce themselves to the equations of an epicycloid.

(1) Let $b = \infty$, so that the source of light is at an infinite distance, and we have a system of parallel rays incident parallel to the axis of x. Then

$$\left.\begin{aligned} x &= \frac{a}{4}(3\cos\theta - \cos 3\theta) \\ y &= \frac{a}{4}(3\sin\theta - \sin 3\theta) \end{aligned}\right\}, \tag{70}$$

which are the equations to an epicycloid; see equations (37), Art. 177, the radii of the fixed and rolling circles being respectively $\frac{a}{2}$ and $\frac{a}{4}$. See fig. 119.

(2) Let $b = a$; in which case the source of light is at the extremity of the diameter of the circle (see fig. 120), and the equations (69) become

$$\left.\begin{aligned} x &= \frac{a}{3}(2\cos\theta - \cos 2\theta) \\ y &= \frac{a}{3}(2\sin\theta - \sin 2\theta) \end{aligned}\right\}, \tag{71}$$

which are the equations to a cardioid (see Art. 178), the radius of the fixed and generating circles being each $\frac{a}{3}$.

274.] Caustics by reflexion from curves expressed in terms of polar coordinates, and which have the origin of light at the pole, may be determined in the following manner; but as the general formulæ are complicated, we will illustrate the method by the particular case of the logarithmic spiral.

In fig. 121 let S be the pole of the spiral and the source of light, SP the incident, PR the reflected ray. Let R be the point in which two successive rays intersect, wherefore R is a point on the caustic; and it is also to be observed that PR is a tangent to the caustic. Let $SP = r$, $SY = p$; $SR = r'$, $SZ = p'$; $PSX = \theta$, $RSX = \theta'$; let the equation to the reflecting curve be

$$r = a^{\theta},$$

and for convenience of writing, let $\log_e a = A$; $\therefore$ by Art. 223, Ex. 3,

$$\tan SPN = \frac{dr}{r d\theta} = A, \tag{72}$$

and

$$r = (1 + A^2)^{\frac{1}{2}} p. \tag{73}$$

From the geometry

$$\frac{p'}{r} = \sin SPZ,$$

$$= \sin 2SPN,$$

$$= \sin(2\tan^{-1} A),$$

$$= \frac{2A}{1 + A^2};$$

$$\therefore \quad p' = \frac{2Ar}{1 + A^2}. \tag{74}$$

Also since

$$SRP + RPS + PSR = 180°,$$

$$\therefore \quad \sin^{-1}\frac{p'}{r'} + 2\tan^{-1} A + \theta' - \theta = 180°;$$

differentiating which, p' and θ varying, but r' and θ' being constant,

$$\frac{dp'}{(r'^2 - p'^2)^{\frac{1}{2}}} - d\theta = 0;$$

$\therefore$ from (72) and (74),

$$\frac{2\text{A}\,dr}{1+\text{A}^2}\frac{1}{(r'^2-p'^2)^{\frac{1}{2}}} = \frac{dr}{\text{A}r} = \frac{2\,dr}{(1+\text{A}^2)\,p'};$$

$$\therefore \quad r' = (1+\text{A}^2)p',$$

which is the equation to a logarithmic spiral, equal to the original one.

275.] We proceed now to consider some of the more general properties of Caustics by Refraction.

Let a, b be the coordinates to the source of light; x, y those to the point on the surface at which the ray is incident; ξ, η those to the refracted ray, and therefore to a point on the caustic; $\mu =$ the refractive index, that is, the ratio of the sine of the angle of incidence to that of the angle of refraction. Let $y = f(x)$ be the equation to the section by the paper of the bounding surface of the refractive medium, the surface being perpendicular to the paper; let r and r' be the distances of the point of incidence from the source of light, and from the point of the caustic; then $\frac{dx}{ds}, \frac{dy}{ds}$ are the cosines of the angles between the tangent to the curve and the coordinate axes: $\frac{x-a}{r}, \frac{y-b}{r}$ those of the angles between the incident ray and the coordinate axes; $\frac{\xi-x}{r'}, \frac{\eta-y}{r'}$ those of the angles between the refracted ray and the coordinate axes; therefore

$$\frac{(x-a)\,dx+(y-b)\,dy}{r\,ds} = \text{sine of angle of incidence},$$

$$\frac{(\xi-x)\,dx+(\eta-y)\,dy}{r'ds} = \text{sine of angle of refraction};$$

therefore by law of refraction,

$$\frac{(x-a)\,dx+(y-b)\,dy}{r\,ds} = \mu\,\frac{(\xi-x)\,dx+(\eta-y)\,dy}{r'ds}; \quad (75)$$

which is the equation to the refracted ray, and of which ξ and η are the current coordinates; the envelope therefore may be found by eliminating x and y between the equation to the refracting curve, the equation (75), and its differential formed by making x and y to vary.

Also since

$$\left. \begin{aligned} r^2 &= (x-a)^2 + (y-b)^2 \\ r'^2 &= (\xi - x)^2 + (\eta - y)^2 \end{aligned} \right\}, \tag{76}$$

(75) becomes

$$dr + \mu\, dr' = 0, \tag{77}$$

therefore $r + \mu r'$ is a constant, or a maximum, or a minimum; but it cannot be a maximum, for such a value would be inconsistent with the geometrical possibility of the problem: therefore it is in general a minimum, and may in certain cases be constant.

Hence also we may prove that all caustics by refraction are *rectifiable* (see Art. 244).

Let ξ, η be the current coordinates and $d\sigma$ the length of the element of the curve of the caustic, so that $d\sigma^2 = d\eta^2 + d\xi^2$; therefore $\frac{d\xi}{d\sigma}$, $\frac{d\eta}{d\sigma}$ are the cosines of the angles made by its tangent with the coordinate axes;

$$\therefore \quad \frac{dy}{d\sigma} = \frac{\eta - y}{r'}, \qquad \frac{d\xi}{d\sigma} = \frac{\xi - x}{r'},$$

$$\therefore \quad \text{from (75)} \qquad dr = \mu \frac{d\xi\, dx + d\eta\, dy}{d\sigma}; \tag{78}$$

and differentiating the latter of (76),

$$r'\, dr' = (\xi - x)(d\xi - dx) + (\eta - y)(d\eta - dy),$$

$$\begin{aligned} \therefore \quad dr' &= \frac{d\xi}{d\sigma}(d\xi - dx) + \frac{d\eta}{d\sigma}(d\eta - dy), \\ &= d\sigma - \frac{d\xi\, dx + d\eta\, dy}{d\sigma}, \\ &= d\sigma - \frac{dr}{\mu}; \end{aligned}$$

$$\therefore \quad \mu\, d\sigma = dr + \mu\, dr',$$

$$\mu\sigma = r + \mu r' + c. \tag{79}$$

An expression exactly analogous to that of Art. 243, and to which therefore a similar mode of explanation is applicable; and therefore the length of the caustic curve is equal to that of two straight lines increased by a constant which is to be determined by the data of the particular problem; but in all cases, if $\sigma_1\, r_1\, r_1'$, $\sigma_2\, r_2\, r_2'$, represent two sets of corresponding values, then

$$\mu(\sigma_2 - \sigma_1) = r_2 - r_1 + \mu(r_2' - r_1').$$

The law of refraction becomes that of reflexion, if $\mu = -1$; and therefore the properties of caustics by refraction proved above are likewise true of caustics by reflexion; attention must however be paid to an ambiguity of sign, of which no notice has been taken in the preceding investigation.

276.] To determine the Caustic by Refraction of Rays refracted at a plane surface; see fig. 122.

Let S be the source of light; SP the incident ray; RPr the refracted ray; $\text{AS} = a$, $\text{AP} = y$; therefore the equation to the refracted ray is

$$\eta - y = -\tan \text{PTA}.\xi,$$

T being the point where PR intersects ASC.

And since

$$\sin \text{SPN} = \mu \sin r\text{P}n,$$

$$\sin \text{PSA} = \mu \sin \text{PTA},$$

$$\frac{y}{(a^2+y^2)^{\frac{1}{2}}} = \mu \sin \text{PTA},$$

$$\therefore \quad \tan \text{PTA} = \frac{y}{\{\mu^2 a^2 + (\mu^2-1)y^2\}^{\frac{1}{2}}};$$

$\therefore$ the equation to PR is

$$\eta - y = \frac{-y}{\{\mu^2 a^2 + (\mu^2-1)y^2\}^{\frac{1}{2}}}\,\xi. \tag{80}$$

Differentiating which with respect to y, and reducing

$$\mu^2 a^2 + (\mu^2-1)\,y^2 = \xi^{\frac{2}{3}}\,\mu^{\frac{4}{3}}\,a^{\frac{4}{3}},$$

$$y = \pm\frac{\mu^{\frac{2}{3}}\,a^{\frac{2}{3}}}{(\mu^2-1)^{\frac{1}{2}}}\,\{\xi^{\frac{2}{3}} - \mu^{\frac{2}{3}}\,a^{\frac{2}{3}}\}^{\frac{1}{2}}; \tag{81}$$

whence, by elimination and from (80),

$$\xi^{\frac{2}{3}} - (\mu^2-1)^{\frac{1}{3}}\,\eta^{\frac{2}{3}} = \mu^{\frac{2}{3}}\,a^{\frac{2}{3}}. \tag{82}$$

Which is the equation to the evolute of the hyperbola if μ be greater than unity; and is that to the evolute of the ellipse if μ be less than unity.

CHAPTER XIV.

APPLICATION OF THE DIFFERENTIAL CALCULUS TO PROPERTIES OF CURVED SURFACES.

277.] An explanation of the mode of generation and of the equations of such curved surfaces and curves in space as are needed for illustration in this and the following Chapters, requires more room than we can afford to give; but it is the less necessary to introduce it, as the ordinary text-books contain sufficient information. It is however desirable to explain the equations to the straight line and the plane, in the forms which we shall employ, as a familiar knowledge of them is requisite to a due understanding of our processes.

(1) To find the Equations to a straight line in space.

Let ξ, η, ζ be the current coordinates to the straight line; x, y, z the coordinates to a point through which the line passes; λ, μ, ν the *direction-angles* of the line; that is, the angles between a parallel line through the origin and the coordinate axes. And let r be the distance between (x, y, z) and (ξ, η, ζ); then the equations to the line are

$$\frac{\xi - x}{\cos\lambda} = \frac{\eta - y}{\cos\mu} = \frac{\zeta - z}{\cos\nu} = r, \tag{1}$$

the last of the equalities following by reason of Preliminary Theorem I.

If therefore the equations to a straight line are given under the form

$$\frac{\xi - x}{\text{L}} = \frac{\eta - y}{\text{M}} = \frac{\zeta - z}{\text{N}}, \tag{2}$$

each of these equalities is by reason of the same Preliminary Theorem equal to

$$\frac{r}{(\text{L}^2 + \text{M}^2 + \text{N}^2)^{\frac{1}{2}}}; \tag{3}$$

and therefore comparing (1) with (2) and (3),

$$\left.\begin{aligned}
\cos\lambda &= \frac{\xi - x}{r} = \frac{\text{L}}{(\text{L}^2+\text{M}^2+\text{N}^2)^{\frac{1}{2}}}\\
\cos\mu &= \frac{\eta - y}{r} = \frac{\text{M}}{(\text{L}^2+\text{M}^2+\text{N}^2)^{\frac{1}{2}}}\\
\cos\nu &= \frac{\zeta - z}{r} = \frac{\text{N}}{(\text{L}^2+\text{M}^2+\text{N}^2)^{\frac{1}{2}}}
\end{aligned}\right\};\qquad (4)$$

and therefore L, M, N in (2) are proportional to the *direction-cosines* of the line, that is, to the cosines of the angles between the line and the coordinate axes.

(2) To find the Equation to a Plane.

DEF.—A plane is a surface generated by a straight line revolving round another straight line which is at right angles to it.

Let the origin be at the point O in the straight line OQ (fig. 123), round which the perpendicular and generating line QP turns, and let λ, μ, ν be the direction-angles of OQ; let ξ, η, ζ be the current coordinates to any point P in the line QP which is in any position, and let OP $= \rho$, OQ $= \delta$; then the direction-cosines of OP are

$$\frac{\xi}{\rho},\quad \frac{\eta}{\rho},\quad \frac{\zeta}{\rho};$$

and since OQP is a right angle,

$$\text{OQ} = \text{OP}\cos\text{POQ},$$

$$\delta = \rho\left\{\frac{\xi}{\rho}\cos\lambda + \frac{\eta}{\rho}\cos\mu + \frac{\zeta}{\rho}\cos\nu\right\};$$

$$\therefore\quad \xi\cos\lambda + \eta\cos\mu + \zeta\cos\nu = \delta, \qquad (5)$$

and as this relation is true for every point in QP, and in every position of QP, it is, according to our definition, the equation to a plane; λ, μ, ν being the direction-angles of the normal to the plane, and δ the length of the perpendicular from the origin on the plane.

Equation (5) immediately follows from the theory of projections, the left-hand side of the equation being the sum of the projections of the *broken* line OMNPQ on the line OQ.

If therefore the equation to a plane be given in the form

$$\text{A}\xi + \text{B}\eta + \text{C}\zeta = \text{D}, \tag{6}$$

on comparing this with (5), we have

$$\frac{\text{A}}{\cos\lambda} = \frac{\text{B}}{\cos\mu} = \frac{\text{C}}{\cos\nu} = \frac{\text{D}}{\delta} = (\text{A}^2 + \text{B}^2 + \text{C}^2)^{\frac{1}{2}};$$

whence it appears that, A, B, C, D are proportional respectively to the direction-cosines of the normal to the plane, and to the length of the perpendicular on the plane from the origin; and we have

$$\left.\begin{aligned} \cos\lambda &= \frac{\text{A}}{(\text{A}^2+\text{B}^2+\text{C}^2)^{\frac{1}{2}}} \\ \cos\mu &= \frac{\text{B}}{(\text{A}^2+\text{B}^2+\text{C}^2)^{\frac{1}{2}}} \\ \cos\nu &= \frac{\text{C}}{(\text{A}^2+\text{B}^2+\text{C}^2)^{\frac{1}{2}}} \\ \delta &= \frac{\text{D}}{(\text{A}^2+\text{B}^2+\text{C}^2)^{\frac{1}{2}}} \end{aligned}\right\}. \tag{7}$$

278.] To find the Equation to a Tangent Plane to a curved Surface at a given point.

Let the equation to the surface be

$$\text{F}(x, y, z) = 0. \tag{8}$$

Our present object is to shew that if a straight line be drawn through a point on the surface (x, y, z), and through a second point $(x+dx,\ y+dy,\ z+dz)$ infinitesimally near to it, the locus of such *tangent* lines is in general a plane, and is what is called the tangent plane. Of course it is manifest that there is in general an infinity of points $(x+dx,\ y+dy,\ z+dz)$ contiguous to the first point, and therefore there is an infinity of tangent lines.

Let ξ, η, ζ be the current coordinates to one of the tangent lines, and x, y, z the coordinates to the point of contact on the surface, then the equations to the line are

$$\frac{\xi - x}{\text{L}} = \frac{\eta - y}{\text{M}} = \frac{\zeta - z}{\text{N}}; \tag{9}$$

as the line passes through the point $(x+dx,\ y+dy,\ z+dz)$, we have

$$\frac{dx}{\text{L}} = \frac{dy}{\text{M}} = \frac{dz}{\text{N}}; \tag{10}$$

and therefore by division,

$$\frac{\xi-x}{dx} = \frac{\eta-y}{dy} = \frac{\zeta-z}{dz} = \frac{r}{ds}, \tag{11}$$

r being the distance of (x, y, z) from any point (ξ, η, ζ) on the line, and

$$ds = \{dx^2 + dy^2 + dz^2\}^{\frac{1}{2}}; \tag{12}$$

that is, ds is the distance between the two points on the surface through which the line passes; (11) therefore are the equations to any straight line touching a surface at a given point.

But the second point through which the line passes must be on the surface, though it may have any position infinitesimally near to (x, y, z); therefore dx, dy, dz must be consistent with the equation to the surface. If therefore at the point under consideration $\left(\frac{d\text{F}}{dx}\right)$, $\left(\frac{d\text{F}}{dy}\right)$, $\left(\frac{d\text{F}}{dz}\right)$ do not all vanish, then

$$\left(\frac{d\text{F}}{dx}\right) dx + \left(\frac{d\text{F}}{dy}\right) dy + \left(\frac{d\text{F}}{dz}\right) dz = 0. \tag{13}$$

Multiplying the several terms of which by the terms of the equalities (11), we have

$$(\xi-x)\left(\frac{d\text{F}}{dx}\right) + (\eta-y)\left(\frac{d\text{F}}{dy}\right) + (\zeta-z)\left(\frac{d\text{F}}{dz}\right) = 0. \tag{14}$$

Now x, y, z, being the coordinates to the point of contact, are constant for a given point, and so are $\left(\frac{d\text{F}}{dx}\right)$, $\left(\frac{d\text{F}}{dy}\right)$, $\left(\frac{d\text{F}}{dz}\right)$ which are functions of x, y, z, and ξ, η, ζ being the current coordinates of the locus, it follows that (14) is of the same form as (6), and therefore represents a plane; and being the locus of the tangent lines to the surface, represents the tangent plane.

279.] On comparing (14) with (6), and with equations (7), if α, β, γ be the direction-angles of the normal to the plane, and p be the perpendicular on the plane from the origin, we have

$$\left.\begin{aligned}
\cos\alpha &= \frac{\left(\frac{d\text{F}}{dx}\right)}{\left\{\left(\frac{d\text{F}}{dx}\right)^2+\left(\frac{d\text{F}}{dy}\right)^2+\left(\frac{d\text{F}}{dz}\right)^2\right\}^{\frac{1}{2}}},\\
\cos\beta &= \frac{\left(\frac{d\text{F}}{dy}\right)}{\left\{\left(\frac{d\text{F}}{dx}\right)^2+\left(\frac{d\text{F}}{dy}\right)^2+\left(\frac{d\text{F}}{dz}\right)^2\right\}^{\frac{1}{2}}},\\
\cos\gamma &= \frac{\left(\frac{d\text{F}}{dz}\right)}{\left\{\left(\frac{d\text{F}}{dx}\right)^2+\left(\frac{d\text{F}}{dy}\right)^2+\left(\frac{d\text{F}}{dz}\right)^2\right\}^{\frac{1}{2}}},
\end{aligned}\right\} \tag{15}$$

$$p = \frac{x\left(\frac{d\text{F}}{dx}\right)+y\left(\frac{d\text{F}}{dy}\right)+z\left(\frac{d\text{F}}{dz}\right)}{\left\{\left(\frac{d\text{F}}{dx}\right)^2+\left(\frac{d\text{F}}{dy}\right)^2+\left(\frac{d\text{F}}{dz}\right)^2\right\}^{\frac{1}{2}}}; \tag{16}$$

whence it appears that $\left(\frac{d\text{F}}{dx}\right)$, $\left(\frac{d\text{F}}{dy}\right)$, $\left(\frac{d\text{F}}{dz}\right)$ are proportional to the direction-cosines of the normal to the tangent plane.

280.] If the equation to the surface be put in the explicit form

$$z = f(x, y), \tag{17}$$

then

$$\text{F}(x, y, z) = f(x, y) - z = 0;$$

$$\left.\begin{aligned}
\therefore\quad \left(\frac{d\text{F}}{dx}\right) &= \left(\frac{df}{dx}\right) = \left(\frac{dz}{dx}\right),\\
\left(\frac{d\text{F}}{dy}\right) &= \left(\frac{df}{dy}\right) = \left(\frac{dz}{dy}\right),\\
\left(\frac{d\text{F}}{dz}\right) &= -1,
\end{aligned}\right\} \tag{18}$$

in which case the equation to the tangent plane becomes

$$\zeta - z = (\xi - x)\left(\frac{dz}{dx}\right) + (\eta - y)\left(\frac{dz}{dy}\right), \tag{19}$$

and (15) and (16) must be modified accordingly.

281.] If the equation to the surface be an homogeneous function of n dimensions of the form

$$F(x, y, z) = c,$$

then, since by the property of such functions proved in Art. 76,

$$x\left(\frac{dF}{dx}\right) + y\left(\frac{dF}{dy}\right) + z\left(\frac{dF}{dz}\right) = nc, \tag{20}$$

the equation to the tangent plane becomes

$$\xi\left(\frac{dF}{dx}\right) + \eta\left(\frac{dF}{dy}\right) + \zeta\left(\frac{dF}{dz}\right) = nc, \tag{21}$$

and (16) becomes

$$p = \frac{nc}{\left\{\left(\frac{dF}{dx}\right)^2 + \left(\frac{dF}{dy}\right)^2 + \left(\frac{dF}{dz}\right)^2\right\}^{\frac{1}{2}}}. \tag{22}$$

Also if the surface be expressed by an *algebraical* equation of the form

$$F(x, y, z) = u_n + u_{n-1} + \dots\dots + u_1 + u_0 = 0, \tag{23}$$

where u_n, $u_{n-1} \dots\dots u_1$, u_0 are homogeneous functions of n, $(n-1) \dots\dots 1, 0$ dimensions, then by a process exactly similar to that of Art. 188, except that in this case there are three variables, it may be shewn that

$$x\left(\frac{dF}{dx}\right) + y\left(\frac{dF}{dy}\right) + z\left(\frac{dF}{dz}\right) = nu_n + (n-1)\,u_{n-1} + \dots\dots + u_1,$$

$$= -\{u_{n-1} + 2u_{n-2} + \dots\dots + (n-1)\,u_1 + nu_0\},$$

and is therefore of only $(n-1)$ dimensions.

Hence the equation to the tangent plane becomes

$$\xi\left(\frac{dF}{dx}\right) + \eta\left(\frac{dF}{dy}\right) + \zeta\left(\frac{dF}{dz}\right)$$

$$= -\{u_{n-1} + 2u_{n-2} + \dots\dots + (n-1)\,u_1 + nu_0\}, \tag{24}$$

and is therefore an equation of only $(n-1)$ dimensions.

In the equation to the tangent plane, considering ξ, η, ζ to be constant, and the coordinates of a given point through which a series of tangent planes is drawn, x, y, z refer to the points of contact on the surface; hence we have the following theorems:

If through a given point planes are drawn, touching a given surface of the nth order, the points of contact lie on a surface of the $(n-1)$th order; and therefore

If through a given point planes are drawn touching a surface of the second order, all the points of contact lie in one plane.

282.] To find the Equation to a Normal of a Curved Surface.

DEF.—A normal is a straight line drawn through any point of a curved surface, and at right angles to the tangent plane at that point.

Let ξ, η, ζ be the current coordinates of the normal, and x, y, z the coordinates to the point where it meets the surface; then, by Art. 279, the direction-cosines of the normal being proportional to $\left(\frac{dF}{dx}\right), \left(\frac{dF}{dy}\right), \left(\frac{dF}{dz}\right)$, its equations are

$$\frac{\xi - x}{\left(\frac{dF}{dx}\right)} = \frac{\eta - y}{\left(\frac{dF}{dy}\right)} = \frac{\zeta - z}{\left(\frac{dF}{dz}\right)}. \tag{25}$$

Also the form of the equations to the normal shews that it is the longest or the shortest line which can be drawn from any point to the surface.

If the equation to the surface be given in the explicit form, these equations, by means of Art. 280, become

$$\left.\begin{aligned} \xi - x &= -\left(\frac{dz}{dx}\right)(\zeta - z) \\ \eta - y &= -\left(\frac{dz}{dy}\right)(\zeta - z) \end{aligned}\right\}. \tag{26}$$

In these equations, if ξ, η, ζ be constant, x, y, z refer to the points on a surface where normals drawn through a given point meet it, and the equations (25) or (26) are those to a curve in space which is the locus of such points of contact.

283.] From (25) it follows, that the equations to a line passing through the origin, and at right angles to the tangent plane, are

$$\frac{\xi}{\left(\frac{dF}{dx}\right)} = \frac{\eta}{\left(\frac{dF}{dy}\right)} = \frac{\zeta}{\left(\frac{dF}{dz}\right)}. \tag{27}$$

By means of which equations, combined with those to the tangent plane and to the surface, may be determined the equation to the surface, which is the locus of the point of intersection of a tangent plane, with the perpendicular drawn to it from the origin.

284.] Examples illustrative of the preceding Articles.

Ex. 1. The Ellipsoid whose equation is

$$\text{F}(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \tag{28}$$

$$\left(\frac{d\text{F}}{dx}\right) = \frac{2x}{a^2}, \qquad \left(\frac{d\text{F}}{dy}\right) = \frac{2y}{b^2}, \qquad \left(\frac{d\text{F}}{dz}\right) = \frac{2z}{c^2};$$

$\therefore$ by equation (14) the equation to the tangent plane is

$$\frac{\xi x}{a^2} + \frac{\eta y}{b^2} + \frac{\zeta z}{c^2} = 1, \tag{29}$$

which also is plainly the equation from (20), since the equation to the surface is an homogeneous function of two dimensions.

Also if α, β, γ are the coordinates to a point through which tangent planes are drawn to an ellipsoid, the equation to the plane of contact is

$$\frac{\alpha x}{a^2} + \frac{\beta y}{b^2} + \frac{\gamma z}{c^2} = 1. \tag{30}$$

Hence also we have by (22),

$$\frac{1}{p} = \pm \left\{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right\}^{\frac{1}{2}}; \tag{31}$$

and the equations to the normal are

$$\frac{a^2}{x}(\xi - x) = \frac{b^2}{y}(\eta - y) = \frac{c^2}{z}(\zeta - z). \tag{32}$$

The equations therefore to a line through the origin, and perpendicular to the tangent plane, are

$$\frac{a^2\xi}{x} = \frac{b^2\eta}{y} = \frac{c^2\zeta}{z}. \tag{33}$$

Whence may be found the equation to the surface, which is the locus of the point of intersection of these lines with the tangent planes;

For ξ, η, ζ being the same in (29) and (33), we have

$$\frac{a^2\xi}{x} = \frac{b^2\eta}{y} = \frac{c^2\zeta}{z} = \frac{a\xi}{\frac{x}{a}} = \frac{b\eta}{\frac{y}{b}} = \frac{c\zeta}{\frac{z}{c}} = \{a^2\xi^2+b^2\eta^2+c^2\zeta^2\}^{\frac{1}{2}}$$

$$= \frac{\xi^2}{\frac{\xi x}{a^2}} = \frac{\eta^2}{\frac{\eta y}{b^2}} = \frac{\zeta^2}{\frac{\zeta z}{c^2}} = \xi^2 + \eta^2 + \zeta^2,$$

$$\therefore \quad \{\xi^2 + \eta^2 + \zeta^2\}^2 = a^2\xi^2 + b^2\eta^2 + c^2\zeta^2; \tag{34}$$

which is the equation to the surface required.

Ex. 2. The Elliptic Paraboloid whose equation is

$$x - \frac{y^2}{4a} - \frac{z^2}{4b} = 0,$$

$$\left(\frac{dF}{dx}\right) = 1, \qquad \left(\frac{dF}{dy}\right) = -\frac{y}{2a}, \qquad \left(\frac{dF}{dz}\right) = -\frac{z}{2b};$$

therefore the equation to the tangent plane is

$$\xi + x - \frac{\eta y}{2a} - \frac{\zeta z}{2b} = 0, \tag{35}$$

and the equations to the line through the origin, and perpendicular to the tangent plane, are

$$\xi = -\frac{2a\eta}{y} = -\frac{2b\zeta}{z}. \tag{36}$$

The equation therefore to the locus of the point of intersection of (36) with (35) is

$$a\eta^2 + b\zeta^2 + \xi(\xi^2 + \eta^2 + \zeta^2) = 0. \tag{37}$$

Ex. 3. If the equation to the surface be

$$xyz = k^3,$$

$$\left(\frac{dF}{dx}\right) = yz = \frac{k^3}{x}, \qquad \left(\frac{dF}{dy}\right) = \frac{k^3}{y}, \qquad \left(\frac{dF}{dz}\right) = \frac{k^3}{z}; \tag{38}$$

and therefore the equation to the tangent plane is

$$(\xi - x)\frac{k^3}{x} + (\eta - y)\frac{k^3}{y} + (\zeta - z)\frac{k^3}{z} = 0,$$

$$\frac{\xi}{x} + \frac{\eta}{y} + \frac{\zeta}{z} = 3; \tag{39}$$

therefore the intercepts of the coordinate axes by the tangent planes are (according to the notation of Art. 186),

$$\xi_0 = 3x, \qquad \eta_0 = 3y, \qquad \zeta_0 = 3z;$$

$$\therefore \quad \xi_0 \, \eta_0 \, \zeta_0 = 27\, xyz,$$

$$= 27\, k^3,$$

that is, the volume of the pyramid contained between the tangent plane and the coordinate planes is constant.

The equations to the line through the origin, and perpendicular to the tangent plane, are

$$\xi x = \eta y = \zeta z = m(\xi \eta \zeta)^{\frac{1}{3}}; \tag{40}$$

$\therefore$ the equation to the locus of the point of intersection of (40) with (39) is

$$\xi^2 + \eta^2 + \zeta^2 = 3m(\xi \eta \zeta)^{\frac{1}{3}}. \tag{41}$$

285.] If at the point on the surface at which the tangent lines of equation (11) are drawn, $\left(\frac{dF}{dx}\right)$, $\left(\frac{dF}{dy}\right)$, and $\left(\frac{dF}{dz}\right)$ all vanish, equation (13) is satisfied independently of any connexion between dx, dy and dz, and therefore does not give a relation whereby to eliminate them; in fact the direction-cosines of the normal at the point are indeterminate, and the tangent plane has no definite position. At such a point therefore there will be a locus of tangent planes, to determine which we must seek for some other relation between dx, dy and dz, arising out of the equation to the surface. Such we have, if all the differential-coefficients of the second order do not vanish at the point in question, in the differential of (13), and which is also the third term of the expansion of $F(x+dx,\ y+dy,\ z+dz)$ in Art. 123, viz.:

$$\left(\frac{d^2F}{dx^2}\right) dx^2 + \left(\frac{d^2F}{dy^2}\right) dy^2 + \left(\frac{d^2F}{dz^2}\right) dz^2 + 2\left(\frac{d^2F}{dy\,dz}\right) dy\,dz$$

$$+ 2\left(\frac{d^2F}{dz\,dx}\right) dz\,dx + 2\left(\frac{d^2F}{dx\,dy}\right) dx\,dy = 0;$$

and multiplying through by corresponding terms of equality (11), we have

$$\left(\frac{d^2F}{dx^2}\right)(\xi-x)^2+\left(\frac{d^2F}{dy^2}\right)(\eta-y)^2+\left(\frac{d^2F}{dz^2}\right)(\zeta-z)^2$$
$$+2\left(\frac{d^2F}{dy\,dz}\right)(\eta-y)(\zeta-z)+2\left(\frac{d^2F}{dz\,dx}\right)(\zeta-z)(\xi-x)$$
$$+2\left(\frac{d^2F}{dx\,dy}\right)(\xi-x)(\eta-y)=0; \tag{42}$$

an equation of the second degree, showing therefore that the locus of the tangent lines is not a plane, but a surface of the second order.

Changing the origin to the point under consideration, the equation assumes the form

$$A\xi^2+B\eta^2+C\zeta^2+2D\eta\zeta+2E\zeta\xi+2F\xi\eta=0, \tag{43}$$

which represents a cone of the second degree, the vertex being at the point of contact; and it may happen that the coefficients have such relations that the equation is decomposable into two factors of the first degree, in which case it will represent two planes.

Ex. 1. Determine the nature of the point at the origin of the surface, whose equation is [see equation (37)]

$$ay^2+bz^2+x(x^2+y^2+z^2)=0;$$

$$\left.\begin{aligned}\left(\frac{dF}{dx}\right)&=3x^2+y^2+z^2=0\\ \left(\frac{dF}{dy}\right)&=2ay+2xy\quad=0\\ \left(\frac{dF}{dz}\right)&=2bz+2xz\quad=0\end{aligned}\right\}\text{ at the origin,}$$

$$\left(\frac{d^2F}{dx^2}\right)=6x=0,\ \left(\frac{d^2F}{dy^2}\right)=2a+2x=2a,\ \left(\frac{d^2F}{dz^2}\right)=2b+2x=2b,$$

$$\left(\frac{d^2F}{dy\,dz}\right)=0,\quad \left(\frac{d^2F}{dz\,dx}\right)=2z=0,\quad \left(\frac{d^2F}{dx\,dy}\right)=2y=0;$$

$\therefore$ equation (43) becomes

$$ay^2+bz^2=0, \tag{44}$$

which is satisfied only by $y=0$, $z=0$; therefore (44) represents the axis of x, or the surface at the origin degenerates into a cuspal point formed round the axis of x.

Ex. 2. A surface is formed by the revolution of a parabola about an ordinate through its focus; it is required to find the nature of the points where it meets the axis of z.

The equation to the surface is

$$16m^2(x^2+y^2)-(z^2-4m^2)^2 = 0,$$

whence it appears that $x = y = 0$, when $z = \pm 2m$; and at such a point $\left(\frac{dF}{dx}\right) = \left(\frac{dF}{dy}\right) = \left(\frac{dF}{dz}\right) = 0$; and differentiating again, and substituting in equation (42), we shall find

$$\xi^2+\eta^2-(\zeta \pm 2m)^2 = 0,$$

the equation to a right-angled circular cone, whose axis is the axis of z and vertex at a distance $\pm 2m$ from the origin.

286.] If all the second differential-coefficients vanish at the point where the tangent plane is to be drawn, we must proceed to a third differentiation, or to the fourth term of the expansion in Art. 123; and thence, in a manner similar to that of the last Article, we shall arrive at a cone of the third order.

CHAPTER XV.

APPLICATION OF THE DIFFERENTIAL CALCULUS TO PROPERTIES OF CURVES IN SPACE.

287.] Thus far we have inquired into the properties of curves which lie wholly in one plane; that is, all their elements and all their consecutive points have been entirely in the plane of xy, and in reference to two fixed lines in that plane we have considered them. It is manifest however that all curves are not subject to the restriction of having their elements in the same plane; there may be non-plane as well as plane curves, and as such they exist in space, and are conveniently referred to three coordinate axes meeting each other at right-angles and in one point; such are also called curves of double curvature, and for a reason which will be hereafter assigned. They may be determined in two ways: either by the intersection of two surfaces whose equations involving x, y, z are given, and therefore by the combination of these two equations; or, what amounts to the same thing, one of the variables, as e. g. z, may have been eliminated between these two equations, and an equation obtained involving only x and y, which will be the equation to the projection of the curve on the plane of xy; and so with the other variables; whereby three equations may be formed, each containing two variables, which will severally represent the projections of the curve on the coordinate planes, and any two of which equations will be sufficient to define the curve; and according as one or the other method is adopted, the formulæ will assume different, though equivalent, shapes.

288.] To find the Equations to a Tangent Line to a Curve in Space.

Def.—A *tangent line* is the straight line passing through two points on the curve which are infinitesimally near to each other.

Let ξ, η, ζ be the current coordinates to the tangent line, and first let the two points through which the line is to pass be at a finite distance Δs apart; and let their coordinates be

$$x, y, z, \qquad x+\Delta x, \; y+\Delta y, \; z+\Delta z;$$

then the equations to the line are

$$\frac{\xi - x}{\Delta x} = \frac{\eta - y}{\Delta y} = \frac{\zeta - z}{\Delta z} = \frac{r}{\Delta s}, \tag{1}$$

where r is the distance between the two points (x, y, z) and (ξ, η, ζ). And when these two points become infinitesimally near to one another, the secant becomes a tangent, and its equations become

$$\frac{\xi - x}{dx} = \frac{\eta - y}{dy} = \frac{\zeta - z}{dz} = \frac{r}{ds}; \tag{2}$$

where $\qquad ds = (dx^2 + dy^2 + dz^2)^{\frac{1}{2}},$

and is the differential of the arc of the curve.

On comparing these equations with those of (4) in Art. 277, if λ, μ, ν be the direction-angles of the tangent,

$$\cos\lambda = \frac{dx}{ds}, \quad \cos\mu = \frac{dy}{ds}, \quad \cos\nu = \frac{dz}{ds}. \tag{3}$$

If then the equations to the curve are two equations, say of the forms $\qquad f(x, z) = 0, \qquad \phi(y, z) = 0,$

$\frac{dx}{dz}$ and $\frac{dy}{dz}$ can be found by differentiation, and equations (2) and (3) can be determined for the particular curve.

And if the curve be determined by means of the equations to two surfaces of the forms

$$\text{F}_1(x, y, z) = 0, \qquad \text{F}_2(x, y, z) = 0, \tag{4}$$

then, since

$$\left.\begin{aligned} \left(\frac{d\text{F}_1}{dx}\right) dx + \left(\frac{d\text{F}_1}{dy}\right) dy + \left(\frac{d\text{F}_1}{dz}\right) dz = 0, \\ \left(\frac{d\text{F}_2}{dx}\right) dx + \left(\frac{d\text{F}_2}{dy}\right) dy + \left(\frac{d\text{F}_2}{dz}\right) dz = 0, \end{aligned}\right\} \tag{5}$$

we have by elimination the following system of equations,

$$\frac{dx}{\left(\frac{d\text{F}_1}{dy}\right)\left(\frac{d\text{F}_2}{dz}\right)-\left(\frac{d\text{F}_1}{dz}\right)\left(\frac{d\text{F}_2}{dy}\right)} = \frac{dy}{\left(\frac{d\text{F}_1}{dz}\right)\left(\frac{d\text{F}_2}{dx}\right)-\left(\frac{d\text{F}_1}{dx}\right)\left(\frac{d\text{F}_2}{dz}\right)}$$

$$= \frac{dz}{\left(\frac{d\text{F}_1}{dx}\right)\left(\frac{d\text{F}_2}{dy}\right)-\left(\frac{d\text{F}_1}{dy}\right)\left(\frac{d\text{F}_2}{dx}\right)}; \tag{6}$$

whence, multiplying the several terms of equality (2) by the several terms of this equality, dx, dy, dz will divide out, and we shall have the equations to the tangent in terms of the partial differential-coefficients of the intersecting surfaces. Similarly may the direction cosines in (3) be determined.

289.] To find the Equation to the Normal Plane to a Curve in Space.

Def.—The plane perpendicular to the tangent line, and passing through the point of contact, is called the *normal plane.*

Let ξ, η, ζ be its current coordinates, and x, y, z be the point of contact through which it passes; then, since it is to be perpendicular to the line whose direction cosines are $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$, its equation is

$$(\xi - x)\,dx + (\eta - y)\,dy + (\zeta - z)\,dz = 0. \tag{7}$$

290.] To find the Equation to the Osculating Plane to a Curve in Space.

In curves such as we have discussed in previous Chapters, all the points lie in one plane, and therefore they are called *plane curves.* This property however does not hold good for all curves in space; although every three consecutive points must be in one plane, yet the fourth may be out of it; or in other words, every two consecutive tangents are in the same plane, but the next consecutive tangent is in general in a different one; our object is to determine the equation to the plane which contains two consecutive tangents, and which is called the *osculating plane,* and is defined as follows:

Def.—The *osculating plane* is the plane containing three consecutive points on a curve.

Let the equation to the plane be

$$\text{A}\xi + \text{B}\eta + \text{C}\zeta = \text{D}, \tag{8}$$

and let it pass through the points on the curve

$$x, y, z,$$
$$x + dx, y + dy, z + dz,$$
$$x + 2dx + d^2x, y + 2dy + d^2y, z + 2dz + d^2z;$$

whence we have

$$\text{A}x + \text{B}y + \text{C}z = \text{D}, \tag{9}$$

$$\text{A}dx + \text{B}dy + \text{C}dz = 0, \tag{10}$$

$$\text{A}d^2x + \text{B}d^2y + \text{C}d^2z = 0; \tag{11}$$

whence, subtracting (9) from (8),

$$\text{A}(\xi - x) + \text{B}(\eta - y) + \text{C}(\zeta - z) = 0; \tag{12}$$

and eliminating successively between (10) and (11) we have the system

$$\frac{\text{A}}{dy\,d^2z - dz\,d^2y} = \frac{\text{B}}{dz\,d^2x - dx\,d^2z} = \frac{\text{C}}{dx\,d^2y - dy\,d^2x}; \tag{13}$$

whence, dividing (12) by the several terms of equality (13), we have

$$(dy\,d^2z - dz\,d^2y)(\xi - x) + (dz\,d^2x - dx\,d^2z)(\eta - y)$$
$$+ (dx\,d^2y - dy\,d^2x)(\zeta - z) = 0, \tag{14}$$

which is the equation to the osculating plane.

291.] The method by which we have deduced this equation is the same as if we had defined the osculating plane to be that in which two consecutive tangents lie, as will be apparent from what follows.

Let the equation to the plane passing through (x, y, z) be

$$\text{A}(\xi - x) + \text{B}(\eta - y) + \text{C}(\zeta - z) = 0;$$

and since it is to be that in which two consecutive tangents lie, whose direction cosines are respectively

$$\frac{dx}{ds}, \quad \frac{dy}{ds}, \quad \frac{dz}{ds},$$

$$\frac{dx + d^2x}{ds + d^2s}, \quad \frac{dy + d^2y}{ds + d^2s}, \quad \frac{dz + d^2z}{ds + d^2s},$$

we must have the conditions

$$A\,dx + B\,dy + C\,dz = 0,$$

$$A(dx + d^2x) + B(dy + d^2y) + C(dz + d^2z) = 0;$$

whence, by subtraction,

$$A\,d^2x + B\,d^2y + C\,d^2z = 0;$$

which two relations between A, B, C are the same as those above marked (10) and (11), whence equality (13) follows, and therefore the equation to the osculating plane is the same.

292.] It is manifest from (7) that all straight lines passing through a point of contact, and perpendicular to the tangent line, lie in the normal plane; two of these normal lines have peculiar properties in relation to the osculating planes, viz. that which is perpendicular to it, and that which lies in it, and is therefore the line of intersection of it by the normal plane. The latter is called the *principal normal*, and the former has the distinctive name of *binormal*, being, as it is, perpendicular to two consecutive elements of the curve, while all other normals are perpendicular to only one.

To find the Equations to the Binormal.

Let l, m, n be its direction-angles; then, as it is perpendicular to the osculating plane,

$$\frac{\cos l}{dy\,d^2z - dz\,d^2y} = \frac{\cos m}{dz\,d^2x - dx\,d^2z} = \frac{\cos n}{dx\,d^2y - dy\,d^2x}$$

$$= \frac{1}{\{(dy\,d^2z - dz\,d^2y)^2 + (dz\,d^2x - dx\,d^2z)^2 + (dx\,d^2y - dy\,d^2x)^2\}^{\frac{1}{2}}}, \tag{15}$$

The denominator of which last expression may be modified as follows:

$$\begin{aligned}(dy\,d^2z - dz\,d^2y)^2 + (dz\,d^2x - dx\,d^2z)^2 + (dx\,d^2y - dy\,d^2x)^2 \\ = (dx^2 + dy^2 + dz^2)\,\{(d^2x)^2 + (d^2y)^2 + (d^2z)^2\} \\ - (dx\,d^2x + dy\,d^2y + dz\,d^2z)^2;\end{aligned} \tag{16}$$

but since
$$ds^2 = dx^2 + dy^2 + dz^2, \tag{17}$$

$$\therefore\quad ds\,d^2s = dx\,d^2x + dy\,d^2y + dz\,d^2z, \tag{18}$$

and therefore the right-hand member of (16) becomes

$$ds^2\{(d^2x)^2 + (d^2y)^2 + (d^2z)^2 - (d^2s)^2\}; \tag{19}$$

and if s becomes equicrescent,

$$ds^2\{(d^2x)^2+(d^2y)^2+(d^2z)^2\}; \tag{20}$$

whence

$$\frac{\cos l}{dy\,d^2z-dz\,d^2y}=\frac{\cos m}{dz\,d^2x-dx\,d^2z}=\frac{\cos n}{dx\,d^2y-dy\,d^2x}$$

$$=\frac{1}{ds\{(d^2x)^2+(d^2y)^2+(d^2z)^2-(d^2s)^2\}^{\frac{1}{2}}}. \tag{21}$$

293.] If z be equicrescent so that $d^2z=0$, then the equation to the osculating plane becomes, dividing through by dz^3,

$$-\frac{d^2y}{dz^2}(\xi-x)+\frac{d^2x}{dz^2}(\eta-y)+\left(\frac{dx}{dz}\frac{d^2y}{dz^2}-\frac{dy}{dz}\frac{d^2x}{dz^2}\right)(\zeta-z)=0. \tag{22}$$

And similarly will the equation (14) be modified, if any other variable be equicrescent.

294.] To find the Equations to the Principal Normal.

Let its equations be

$$\frac{\xi-x}{\text{L}}=\frac{\eta-y}{\text{M}}=\frac{\zeta-z}{\text{N}}; \tag{23}$$

then, by reason of its being perpendicular to the tangent line, and of its lying in the osculating plane, we have

$$\text{L}\,dx+\text{M}\,dy+\text{N}\,dz=0, \tag{24}$$

$$\text{L}(dy\,d^2z-dz\,d^2y)+\text{M}(dz\,d^2x-dx\,d^2z)+\text{N}(dx\,d^2y-dy\,d^2x)=0; \tag{25}$$

whence we have

$$\frac{\text{L}}{dy\,(dx\,d^2y-dy\,d^2x)-dz\,(dz\,d^2x-dx\,d^2z)}=\ldots\ldots$$

or

$$\frac{\text{L}}{dx\,(dx\,d^2x+dy\,d^2y+dz\,d^2z)-d^2x\,(dx^2+dy^2+dz^2)}=\ldots\ldots$$

and since

$$ds^2=dx^2+dy^2+dz^2, \tag{26}$$

$$ds\,d^2s=dx\,d^2x+dy\,d^2y+dz\,d^2z, \tag{27}$$

we have

$$\frac{\text{L}}{dx\,ds\,d^2s-ds^2\,d^2x}=\ldots\ldots$$

or

$$\frac{\text{L}}{d\left(\frac{dx}{ds}\right)}=\frac{\text{M}}{d\left(\frac{dy}{ds}\right)}=\frac{\text{N}}{d\left(\frac{dz}{ds}\right)}; \tag{28}$$

and therefore the equations to the principal normal are

$$\frac{\xi-x}{d\left(\frac{dx}{ds}\right)} = \frac{\eta-y}{d\left(\frac{dy}{ds}\right)} = \frac{\zeta-z}{d\left(\frac{dz}{ds}\right)}; \tag{29}$$

and if s be equicrescent,

$$\frac{\xi-x}{d^2x} = \frac{\eta-y}{d^2y} = \frac{\zeta-z}{d^2z}. \tag{30}$$

Therefore, if λ, μ, ν are the direction-angles of the principal normal, we have

$$\frac{\cos\lambda}{d\left(\frac{dx}{ds}\right)} = \frac{\cos\mu}{d\left(\frac{dy}{ds}\right)} = \frac{\cos\nu}{d\left(\frac{dz}{ds}\right)}$$

$$= \frac{ds}{\{(d^2x)^2+(d^2y)^2+(d^2z)^2-(d^2s)^2\}^{\frac{1}{2}}}, \tag{31}$$

as will be found on reduction.

295.] Examples on the preceding.

Ex. 1. The curve formed by the intersection of an Ellipsoid by a Plane.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1,$$

$$\text{A}x+\text{B}y+\text{C}z = 0;$$

$$\frac{x}{a^2}\,dx+\frac{y}{b^2}\,dy+\frac{z}{c^2}\,dz = 0,$$

$$\text{A}\,dx+\text{B}\,dy+\text{C}\,dz = 0;$$

$$\therefore\quad \frac{dx}{\text{C}\frac{y}{b^2}-\text{B}\frac{z}{c^2}} = \frac{dy}{\text{A}\frac{z}{c^2}-\text{C}\frac{x}{a^2}} = \frac{dz}{\text{B}\frac{x}{a^2}-\text{A}\frac{y}{b^2}};$$

therefore the equations to the tangent line are

$$\therefore\quad \frac{\xi-x}{\text{C}\frac{y}{b^2}-\text{B}\frac{z}{c^2}} = \frac{\eta-y}{\text{A}\frac{z}{c^2}-\text{C}\frac{x}{a^2}} = \frac{\zeta-z}{\text{B}\frac{x}{a^2}-\text{A}\frac{y}{b^2}};$$

and the equation to the normal plane is

$$(\xi-x)\left(\mathrm{C}\frac{y}{b^2}-\mathrm{B}\frac{z}{c^2}\right)+(\eta-y)\left(\mathrm{A}\frac{z}{c^2}-\mathrm{C}\frac{x}{a^2}\right)$$
$$+(\zeta-z)\left(\mathrm{B}\frac{x}{a^2}-\mathrm{A}\frac{y}{b^2}\right)=0.$$

Ex. 2. The Helix; see fig. 125.

Let OA = OB = a be the radius of the base-cylinder of the Helix, and ϕ(= AON) be the angle between the plane of xz and the radius of the cylinder drawn to the point whose co-ordinates are x, y, z, and whose projection on the plane of xy is ON; and let OM = x, MN = y, NP = z; and let k be the tangent of the angle at which the thread of the helix is inclined to the plane of xy; so that NP = $k \times$ the arc AN, whereby the equations to the curve are

$$x=a\cos\phi, \qquad y=a\sin\phi, \qquad z=ka\phi; \tag{32}$$

$$\therefore\ \left.\begin{array}{l} dx=-a\sin\phi\, d\phi, \quad dy=a\cos\phi\, d\phi, \quad dz=ka\,d\phi, \\ d^2x=-a\cos\phi\, d\phi^2, \quad d^2y=-a\sin\phi\, d\phi^2, \quad d^2z=0, \end{array}\right\} \tag{33}$$

the differentiations being performed on the supposition that ϕ is equicrescent; therefore the equations to the tangent are

$$\frac{\xi-x}{-a\sin\phi}=\frac{\eta-y}{a\cos\phi}=\frac{\zeta-z}{ka}. \tag{34}$$

The equation to the normal plane is

$$-a(\xi-x)\sin\phi+a(\eta-y)\cos\phi+ka(\zeta-z)=0,$$
$$\eta x-\xi y+ka(\zeta-z)=0; \tag{35}$$

when $\xi=\eta=0$, $\zeta=z$; the normal plane therefore cuts the axis of z at a distance from the origin, equal to the z of the helix at which it is drawn.

Also
$$ds^2=dx^2+dy^2+dz^2,$$
$$=(1+k^2)a^2d\phi^2; \tag{36}$$

therefore if λ, μ, ν are the direction-angles of the tangent

$$\left.\begin{array}{l} \cos\lambda=\dfrac{dx}{ds}=\dfrac{-\sin\phi}{\{1+k^2\}^{\frac{1}{2}}}, \\ \cos\mu=\dfrac{dy}{ds}=\dfrac{\cos\phi}{\{1+k^2\}^{\frac{1}{2}}}, \\ \cos\nu=\dfrac{dz}{ds}=\dfrac{k}{\{1+k^2\}^{\frac{1}{2}}}. \end{array}\right\} \tag{37}$$

The tangent therefore is always inclined at the same angle to the axis of z.

Hence also the equation to the osculating plane is

$$ka^2 \sin\phi(\xi - x) - ka^2 \cos\phi(\eta - y) + a^2(\zeta - z) = 0,$$

or
$$k(\xi y - \eta x) + a(\zeta - z) = 0. \qquad (38)$$

Also from (37) and (36), taking s to be equicrescent,

$$\left.\begin{aligned} \frac{d^2x}{ds^2} &= -\frac{\cos\phi}{(1+k^2)^{\frac{1}{2}}}\frac{d\phi}{ds} = -\frac{\cos\phi}{a(1+k^2)}, \\ \frac{d^2y}{ds^2} &= -\frac{\sin\phi}{(1+k^2)^{\frac{1}{2}}}\frac{d\phi}{ds} = -\frac{\sin\phi}{a(1+k^2)}, \\ \frac{d^2z}{ds^2} &= 0; \end{aligned}\right\} \qquad (39)$$

therefore the direction-cosines of the principal normal are, by reason of (31), $\cos\phi$, $\sin\phi$, and 0. The principal normal is therefore perpendicular to the axis of z, and coincident with the radius of the base-cylinder drawn to the point (x, y, z).

296.] In connexion with the subject of the osculating plane, it is convenient to determine the analytical condition, that a curve in space may be wholly in one plane; or in other words, that every four consecutive points on the curve may be in one plane.

Let the equation to the plane be

$$\text{A}x + \text{B}y + \text{C}z = \text{D};$$

then
$$\left.\begin{aligned} \text{A}dx + \text{B}dy + \text{C}dz &= 0, \\ \text{A}d^2x + \text{B}d^2y + \text{C}d^2z &= 0, \\ \text{A}d^3x + \text{B}d^3y + \text{C}d^3z &= 0; \end{aligned}\right\} \qquad (40)$$

whence, from the last three equations, by cross-multiplication,

$$dx(d^2y\,d^3z - d^2z\,d^3y) + dy(d^2z\,d^3x - d^2x\,d^3z)$$
$$+ dz(d^2x\,d^3y - d^2y\,d^3x) = 0; \qquad (41)$$

which condition becomes, if z be taken to be an equicrescent variable,

$$\frac{d^2x}{dz^2}\frac{d^3y}{dz^3} - \frac{d^2y}{dz^2}\frac{d^3x}{dz^3} = 0, \qquad (42)$$

the geometrical meaning of which condition is explained in Art. 331.

297.] Of lines which can be drawn on a given surface, and which are therefore generally curves of double curvature, one class requires to be alluded to: though, for a full discussion of their properties, more space is needed than can be afforded to it in an elementary treatise.

Let $\text{F}(x, y, z) = c$ be the equation to the surface; lines may manifestly be drawn on it, so that, at the common points, the principal normal may be coincident with the normal to the surface. In which case the differential equations of such lines are

$$\frac{d\left(\frac{dx}{ds}\right)}{\left(\frac{d\text{F}}{dx}\right)} = \frac{d\left(\frac{dy}{ds}\right)}{\left(\frac{d\text{F}}{dy}\right)} = \frac{d\left(\frac{dz}{ds}\right)}{\left(\frac{d\text{F}}{dz}\right)}; \tag{43}$$

such are called *geodesic lines*, and are the shortest that can be drawn on the surface between any two points in the same.

CHAPTER XVI.

ON THE GENERAL AND PARTIAL-DIFFERENTIAL EQUATIONS OF SURFACES, GENERATED BY LINES MOVING ACCORDING TO GIVEN LAWS.

298.]* In the present Chapter I propose to consider a few simple properties of surfaces, which are generated by straight lines and circles moving according to given laws; which lines, as they produce the surface, are called *generators*. The general theory is as follows:

Suppose that we have two equations involving x, y, z and two constants c_1 and c_2, and that they are of the forms

$$\text{F}_1(x, y, z) = c_1, \qquad \text{F}_2(x, y, z) = c_2, \tag{1}$$

each of which represents a surface; and they, when taken conjointly, represent the line of intersection of the two surfaces. But if c_1 and c_2 are variable parameters, and dependent on each other by means of another equation,

$$f(c_1, c_2) = 0, \tag{2}$$

then, as c_1 and c_2 vary, the line of intersection of the two surfaces (1) varies, and by a continuous variation generates a surface, the equation of such a surface being found by the substitution of (1) in (2), whereby we have

$$f(\text{F}_1, \text{F}_2) = 0. \tag{3}$$

The form however which such problems actually assume is generally somewhat different: a geometrical condition is given which is equivalent to the equation (2); thus, for instance, the generator may be a straight line which is to pass through a given curve, and move parallel to itself, or be parallel to a given plane

* Further information on subjects connected with the discussions of the present Chapter will be found in "Application d'Analyse à la Géométrie, par G. Monge; 5me edition, par M. Liouville, Paris, 1850."

and pass through two given lines, in which cases the curves through which the generator has to pass are called *directors*. The process of elimination is as follows:

Let (1) be the equations to the generator involving two *independent* variable parameters, c_1 and c_2; and let the equations to the *director* be

$$\Phi_1(x, y, z) = 0, \qquad \Phi_2(x, y, z) = 0, \tag{4}$$

then, as the generator is to pass through the director, x, y, z are at that common point the same in (1) and (4); eliminating therefore x, y, z (which refer to that common point) between these four equations, there will result a relation between c_1 and c_2 of the same form as (2), in which they must be replaced by their values in (1), and the resulting equation between x, y, z is that to the surface.

Again, the condition to which the generator is to be subject frequently is that it should circumscribe a given surface; the generator therefore must touch the given surface at their common point. Let the equation to the surface which is to be circumscribed be

$$u = 0, \tag{5}$$

the direction-cosines of its normal at any point are proportional to

$$\left(\frac{du}{dx}\right), \qquad \left(\frac{du}{dy}\right), \qquad \left(\frac{du}{dz}\right), \tag{6}$$

and the direction-cosines of the tangent of the generator are, Art. 288, equation (6), proportional to

$$\left.\begin{array}{l} \left(\frac{d\text{F}_1}{dy}\right)\left(\frac{d\text{F}_2}{dz}\right) - \left(\frac{d\text{F}_1}{dz}\right)\left(\frac{d\text{F}_2}{dy}\right), \\ \left(\frac{d\text{F}_1}{dz}\right)\left(\frac{d\text{F}_2}{dx}\right) - \left(\frac{d\text{F}_1}{dx}\right)\left(\frac{d\text{F}_2}{dz}\right), \\ \left(\frac{d\text{F}_1}{dx}\right)\left(\frac{d\text{F}_2}{dy}\right) - \left(\frac{d\text{F}_1}{dy}\right)\left(\frac{d\text{F}_2}{dx}\right), \end{array}\right\} \tag{7}$$

which we will symbolize respectively by $\text{P}, \text{Q}, \text{R}$; and as these lines are to be perpendicular to each other at the point of contact, so that the generator may touch the director-surface, we have the condition

$$\text{P}\left(\frac{du}{dx}\right) + \text{Q}\left(\frac{du}{dy}\right) + \text{R}\left(\frac{du}{dz}\right) = 0, \tag{8}$$

at their common points, which is the equation to the curve of contact; and its intersection with (5) gives us a director through which the generator is to pass. Between therefore (1), (5) and (8), we may eliminate the coordinates which refer to their common points of contact, and get a relation between c_1 and c_2, for which we may substitute the general values of the coordinates given by (1). I propose then to consider those properties of the surfaces which the Differential Calculus enables us to elucidate.

Section 1.—*On Surfaces generated by the motion of Straight Lines.*

299.] Surfaces generated by the motion of straight lines are generally termed *ruled* surfaces (*surfaces réglées*), and of them there are two distinct classes: according as two consecutive generating lines do or do not intersect each other; or in other words, according as two consecutive generators are in the same or in different planes. Surfaces of the former class are termed developable, and those of the latter skew surfaces (*surfaces gauches*).

The equations to a straight line being

$$\frac{x-x'}{\text{L}} = \frac{y-y'}{\text{M}} = \frac{z-z'}{\text{N}}, \tag{9}$$

six constants are apparently involved; of which however only four are indeterminate, because the equations can be put into the forms

$$\left.\begin{aligned} x &= az + a, \\ y &= \beta z + b, \end{aligned}\right\} \tag{10}$$

and of these four variable parameters, two fix the direction of the line and two fix its position. To eliminate these and to find the equation to the surface, five conditions are required, two of which are of necessity the equations (10) of the generator, and the other three are indeterminate, and may be given by means of the equations of three directors. Hence no ruled surface can in general have more than three directors; and to determine the surface from the equations to the generator, such conditions, or others equivalent to them in number, must be given. In general the directors may be such that two con-

secutive generators do not intersect, in which case the surface generated is skew; when however two successive generators intersect, the analytical condition of this being the case satisfies one of the relations which are required amongst the parameters, and leaves only two to be satisfied by the equations of the fixed directors. Developable surfaces cannot therefore in general have more than two directors.

300.] On Developable Surfaces.

Since in developable surfaces every generating line and its consecutive line are in the same plane, this plane is the tangent plane to the surface at every point along the first line: for consider any point on the first generating line; the tangent plane at that point passes through the next consecutive point on the line, and therefore contains the line; and as the tangent plane also passes through an indefinite number of points infinitesimally near to the point at which it is drawn, it also passes through a point on the consecutive generating line; and this line is in the same plane with the first generating line; therefore the tangent plane which contains the first line also contains this latter line; the tangent plane therefore touches the surface along the whole length of the generating line. Of this proposition we shall hereafter have an analytical proof, deduced from the general equation of these surfaces. Hence also we have the following property of such surfaces:

Let any number of generating lines be represented by G_1, G_2, G_3 ......; then the surface is made up of the infinitesimal plane areas contained between G_1 and G_2, between G_2 and G_3 ...... Now the plane area lying between G_1 and G_2 may be brought into the same plane with that lying between G_2 and G_3, by being turned through a small angle about G_2; and similarly, by turning this last area about G_3, may all the areas between G_1 and G_3 be brought into the same plane without any discontinuity. Let these operations be performed for all the elements, then all will be brought into the same plane; and if we suppose any thin flexible and inextensible film to be laid on such a surface, it will be unfolded into a plane without tearing, rumpling, or doubling. For this reason have such surfaces obtained the characteristic title of Developable.

If all the generating lines of such surfaces meet in one point the surface is called *conical*, and the point is called the vertex of the cone; and if that point be at an infinite distance, so that all the generators are parallel, the surface is called *cylindrical*.

301.] On Conical Surfaces.

Let a, b, c be the coordinates to the vertex of the cone; then the equations to the generating line are

$$\frac{x-a}{\text{L}} = \frac{y-b}{\text{M}} = \frac{z-c}{\text{N}}, \tag{11}$$

in which a, b, c are constant; and $\text{L}, \text{M}, \text{N}$, which are constant for any one position of the generator, vary as the generator passes from one position to another.

Let the equations to the director be

$$\text{F}_1(x', y', z') = 0, \qquad \text{F}_2(x', y', z') = 0; \tag{12}$$

and therefore, as the generator has to pass through x', y', z', its equations become

$$\frac{x-a}{x'-a} = \frac{y-b}{y'-b} = \frac{z-c}{z'-c}; \tag{13}$$

whence

$$\left.\begin{aligned} x'-a &= \frac{x-a}{z-c}(z'-c), \\ y'-b &= \frac{y-b}{z-c}(z'-c), \end{aligned}\right\} \tag{14}$$

between which equations and (12), eliminating x', y', z', we have a function of the form

$$\text{F}\left(\frac{x-a}{z-c}, \frac{y-b}{z-c}\right) = 0, \tag{15}$$

which is the general functional equation of conical surfaces.

Equation (15) may also be written in the explicit form

$$\frac{x-a}{z-c} = f\left(\frac{y-b}{z-c}\right). \tag{16}$$

Also the equation may be put into the following symmetrical form

$$\text{F}\left(\frac{y-b}{z-c}, \frac{z-c}{x-a}, \frac{x-a}{y-b}\right) = 0. \tag{17}$$

If the origin be taken at the vertex of the cone, $a = b = c = 0$, and the last three equations severally become

$$\text{F}\left(\frac{x}{z}, \frac{y}{z}\right) = 0,$$

$$\frac{x}{z} = f\left(\frac{y}{z}\right),$$

$$\text{F}\left(\frac{y}{z}, \frac{z}{x}, \frac{x}{y}\right) = 0,$$

which are homogeneous functions of 0 dimensions and equal to 0; such a function therefore is the equation to a conical surface.

302.] To find the Differential Equation to Conical Surfaces.

Eliminate the arbitrary function from (16), as in Ex. 2, Art. 89, and there results

$$(x-a)\left(\frac{dz}{dx}\right) + (y-b)\left(\frac{dz}{dy}\right) = z - c; \qquad (18)$$

or taking equation (17), and representing by F' the derived-function of F, we have

$$\left(\frac{d\text{F}}{dx}\right) = \left\{-\frac{z-c}{(x-a)^2} + \frac{1}{y-b}\right\}\text{F}',$$

$$\left(\frac{d\text{F}}{dy}\right) = \left\{-\frac{x-a}{(y-b)^2} + \frac{1}{z-c}\right\}\text{F}',$$

$$\left(\frac{d\text{F}}{dz}\right) = \left\{-\frac{y-b}{(z-c)^2} + \frac{1}{x-a}\right\}\text{F}';$$

$$\therefore \quad (x-a)\left(\frac{d\text{F}}{dx}\right) + (y-b)\left(\frac{d\text{F}}{dy}\right) + (z-c)\left(\frac{d\text{F}}{dz}\right) = 0; \quad (19)$$

and this, or (18), (which is identical with it, as shewn in Art. 89,) is the general differential equation to conical surfaces.

Equation (19), as appears from Art. 278, is the equation to the tangent plane of the surface; and the form of it shews that it always passes through the vertex, and has therefore a generating line lying in it.

303.] Examples of Conical Surfaces.

Ex. 1. To find the Equation to a Cone whose Director is a Circle in the Plane of xy.

Let the equation to the circle be

$$x_0^2 + y_0^2 = k^2, \tag{20}$$

and the equations to the generator

$$\frac{x-a}{\text{L}} = \frac{y-b}{\text{M}} = \frac{z-c}{\text{N}}; \tag{21}$$

$\therefore$ when $z = 0$, $\quad x_0 = a - \frac{\text{L}}{\text{N}} c,$

$$y_0 = b - \frac{\text{M}}{\text{N}} c;$$

squaring and adding which, by means of (20), and replacing the variable parameters by their values from (21), we have

$$(cx - az)^2 + (cy - bz)^2 = k^2(z - c)^2,$$

which is the general equation to a cone which has a circular director.

If the line joining the centre of the circular director and the vertex be at right angles to the circle, the cone is called Right; in which case $a = b = 0$, and the equation is

$$x^2 + y^2 = \frac{k^2}{c^2}(z - c)^2,$$

where $\frac{k}{c}$ is the tangent of the semi-vertical-angle.

Ex. 2. To find the Equation to a Cone circumscribing a given Ellipsoid.

Let the coordinates to the vertex of the cone be x_0, y_0, z_0, and the equation to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \tag{22}$$

Let the equations to the generator be

$$\frac{\xi - x_0}{\text{L}} = \frac{\eta - y_0}{\text{M}} = \frac{\zeta - z_0}{\text{N}};$$

and as the generators are to touch the ellipsoid, they may be put in the form

$$\frac{\xi-x}{x_0-x} = \frac{\eta-y}{y_0-y} = \frac{\zeta-z}{z_0-z}, \tag{23}$$

and as the points (ξ, η, ζ), (x_0, y_0, z_0) are in the tangent plane to the ellipsoid

$$\frac{\xi x}{a^2} + \frac{\eta y}{b^2} + \frac{\zeta z}{c^2} = 1, \tag{24}$$

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} + \frac{z_0 z}{c^2} = 1; \tag{25}$$

whence, operating on the equality (23), and reducing by means of (22) (24) and (25), we have

$$\frac{\xi-x}{x_0-x} = \frac{\eta-y}{y_0-y} = \frac{\zeta-z}{z_0-z} = \frac{\frac{\xi x_0}{a^2} + \frac{\eta y_0}{b^2} + \frac{\zeta z_0}{c^2} - 1}{\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2} - 1}$$

$$= \frac{\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} + \frac{\zeta^2}{c^2} - 1}{\frac{\xi x_0}{a^2} + \frac{\eta y_0}{b^2} + \frac{\zeta z_0}{c^2} - 1}; \tag{26}$$

$$\therefore \left(\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2} - 1\right)\left(\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} + \frac{\zeta^2}{c^2} - 1\right)$$

$$= \left\{\frac{\xi x_0}{a^2} + \frac{\eta y_0}{b^2} + \frac{\zeta c_0}{c^2} - 1\right\}^2; \tag{27}$$

which is the equation to the circumscribing cone.

As x, y, z refer to the point on the ellipsoid which is common to it and to the cone, (25) is the equation to the plane of contact; and it, and equation (22), are those to the director-curve.

304.] To find the Equation to Cylindrical Surfaces.

As the point through which all the generators of a cylindrical surface pass is at an infinite distance, they all are parallel to each other; let therefore the equations be

$$\frac{x-x'}{\text{L}} = \frac{y-y'}{\text{M}} = \frac{z-z'}{\text{N}}; \tag{28}$$

where x, y, z are the current coordinates to the surface, x', y', z' refer to a point on the director, and L, M, N are constant.

Let the equations to the director be

$$F_1(x', y', z') = 0, \qquad F_2(x', y', z') = 0, \tag{29}$$

and from (28) we have

$$\left.\begin{aligned} x' &= x - \frac{L}{N}(z - z'), \\ y' &= y - \frac{M}{N}(z - z'); \end{aligned}\right\} \tag{30}$$

from which four equations, eliminating x', y', z', we have a result of the form

$$F(Nx - Lz,\ Ny - Mz) = 0, \tag{31}$$

which is the general equation to cylindrical surfaces.

Equation (31) may also be put in the form

$$x - lz = f(y - mz), \tag{32}$$

which is an explicit form of the general equation to cylindrical surfaces.

Also (31) may be written in the following symmetrical form

$$F(Ny - Mz,\ Lz - Nx,\ Mx - Ly) = 0. \tag{33}$$

Suppose that the director is a plane curve in the plane xy, and that its equation is

$$x_0 = f(y_0); \tag{34}$$

then, from (28),

$$\left.\begin{aligned} x_0 &= x - \frac{L}{N}z = x - lz, \\ y_0 &= y - \frac{M}{N}z = y - mz, \end{aligned}\right\} \tag{35}$$

which being substituted in (34) give

$$x - lz = f(y - mz).$$

Suppose however that the cylinder is to circumscribe a surface whose equation is $F_1(x', y', z') = 0$; then, as the generator is to be perpendicular to the normal at the point of contact,

$$L\left(\frac{dF_1}{dx'}\right) + M\left(\frac{dF_1}{dy'}\right) + N\left(\frac{dF_1}{dz'}\right) = 0, \tag{36}$$

from which, the equation to the surface, and the equations to the generators, x', y', z' are to be eliminated, and the resulting equation in terms of x, y, z is that to the cylindrical surface.

305.] To find the Differential Equation to Cylindrical Surfaces.

Eliminating the arbitrary function from (32) by means of differentiation, see Ex. 1, Art. 89, we have

$$l\left(\frac{dz}{dx}\right) + m\left(\frac{dz}{dy}\right) = 1. \qquad (37)$$

Or taking equation (33), and representing its derived-function by F', we have

$$\left(\frac{d\mathrm{F}}{dx}\right) = (\mathrm{M}-\mathrm{N})\,\mathrm{F}', \qquad \left(\frac{d\mathrm{F}}{dy}\right) = (\mathrm{N}-\mathrm{L})\,\mathrm{F}', \qquad \left(\frac{d\mathrm{F}}{dz}\right) = (\mathrm{L}-\mathrm{M})\,\mathrm{F}';$$

$$\therefore\quad \mathrm{L}\left(\frac{d\mathrm{F}}{dx}\right) + \mathrm{M}\left(\frac{d\mathrm{F}}{dy}\right) + \mathrm{N}\left(\frac{d\mathrm{F}}{dz}\right) = 0, \qquad (38)$$

and this, or (37), (which is identical with it, as shewn in Ex. 1, Art. 89,) is the general differential equation to all cylindrical surfaces.

The geometrical meaning of (38) is, that the normal to the surface is perpendicular to the generating line, whose direction-cosines are proportional to L, M, N.

306.] Examples of Cylindrical Surfaces.

Ex. 1. To find the Equation to the Cylinder whose Director is the Ellipse,

$$\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1, \qquad (39)$$

substituting for x_0 and y_0, in terms of (35), we have

$$\frac{(\mathrm{N}x - \mathrm{L}z)^2}{a^2\,\mathrm{N}^2} + \frac{(\mathrm{N}y - \mathrm{M}z)^2}{b^2\,\mathrm{N}^2} = 1, \qquad (40)$$

which is the general equation to an oblique elliptical cylinder. And if the generator is perpendicular to the plane of xy, the cylinder is called Right, and as in that case $\mathrm{L} = 0$, $\mathrm{M} = 0$, the equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Hence the equation to an oblique circular cylinder is

$$(\mathrm{N}x - \mathrm{L}z)^2 + (\mathrm{N}y - \mathrm{M}z)^2 = \mathrm{N}^2 a^2, \qquad (41)$$

Ex. 2. To find the Equation to a Cylinder circumscribing a given Ellipsoid.

Let the equations to the generator be

$$\frac{\xi - x}{\text{L}} = \frac{\eta - y}{\text{M}} = \frac{\zeta - z}{\text{N}}, \tag{42}$$

and that to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1; \tag{43}$$

then, as (42) touches (43), we have

$$\frac{\xi x}{a^2} + \frac{\eta y}{b^2} + \frac{\zeta z}{c^2} = 1; \tag{44}$$

whence, operating on the members of equality (42),

$$\frac{\xi - x}{\text{L}} = \frac{\eta - y}{\text{M}} = \frac{\zeta - z}{\text{N}} = \frac{\dfrac{\xi^2}{a^2} + \dfrac{\eta^2}{b^2} + \dfrac{\zeta^2}{c^2} - 1}{\dfrac{\text{L}\xi}{a^2} + \dfrac{\text{M}\eta}{b^2} + \dfrac{\text{N}\zeta}{c^2}}$$

$$= \frac{\left\{\dfrac{\xi^2}{a^2} + \dfrac{\eta^2}{b^2} + \dfrac{\zeta^2}{c^2} - 1\right\}^{\frac{1}{2}}}{\left\{\dfrac{\text{L}^2}{a^2} + \dfrac{\text{M}^2}{b^2} + \dfrac{\text{N}^2}{c^2}\right\}^{\frac{1}{2}}}; \tag{45}$$

$$\therefore \quad \left(\frac{\text{L}\xi}{a^2} + \frac{\text{M}\eta}{b^2} + \frac{\text{N}\zeta}{c^2}\right)^2$$

$$= \left(\frac{\text{L}^2}{a^2} + \frac{\text{M}^2}{b^2} + \frac{\text{N}^2}{c^2}\right)\left(\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} + \frac{\zeta^2}{c^2} - 1\right). \tag{46}$$

Ex. 3. To determine the conditions that the general Equation of the second degree of three variables may represent a Cylinder.

Let the general equation be

$$\text{F}(x, y, z) = \text{A}x^2 + \text{B}y^2 + \text{C}z^2 + 2\text{A}_1 yz + 2\text{B}_1 zx + 2\text{C}_1 xy$$
$$+ 2\text{A}_2 x + 2\text{B}_2 y + 2\text{C}_2 z + \text{K} = 0,$$

$$\left(\frac{d\text{F}}{dx}\right) = 2(\text{A}x + \text{C}_1 y + \text{B}_1 z + \text{A}_2),$$

$$\left(\frac{d\text{F}}{dy}\right) = \ldots\ldots\ldots \quad \left(\frac{d\text{F}}{dz}\right) = \ldots\ldots\ldots$$

and therefore equation (38) becomes

$$\text{L}(\text{A}x + \text{C}_1 y + \text{B}_1 z + \text{A}_2) + \text{M}(\text{C}_1 x + \text{B}y + \text{A}_1 z + \text{B}_2) \\ + \text{N}(\text{B}_1 x + \text{A}_1 y + \text{C}z + \text{C}_2) = 0;$$

and as this condition is to hold for all values of x, y, z

$$\begin{aligned} \text{AL} + \text{C}_1\text{M} + \text{B}_1\text{N} &= 0, \\ \text{C}_1\text{L} + \text{BM} + \text{A}_1\text{N} &= 0, \\ \text{B}_1\text{L} + \text{A}_1\text{M} + \text{CN} &= 0, \\ \text{A}_2\text{L} + \text{B}_2\text{M} + \text{C}_2\text{N} &= 0; \end{aligned}$$

$\therefore$ from the first three

$$\text{ABC} - \text{AA}_1{}^2 - \text{BB}_1{}^2 - \text{CC}_1{}^2 + 2\text{A}_1\text{B}_1\text{C}_1 = 0,$$

and from the first three combined with the fourth

$$(\text{BC} - \text{A}_1{}^2)\,\text{A}_2 + (\text{CA} - \text{B}_1{}^2)\,\text{B}_2 + (\text{AB} - \text{C}_1{}^2)\,\text{C}_2 = 0.$$

By a similar method may equation (19) be applied to determine the relations amongst the constants, that an equation of the second degree may represent a cone.

307.] On the Equations of Developable Surfaces.

As any and every two consecutive generators of a developable surface intersect, and as these two (as shewn in Art. 300) lie in one plane, it is convenient to consider such a surface as formed by the continual intersection of planes drawn according to a given law. Now the general equation to a plane involves only three independent constants, and it is convenient to put it in the form

$$\text{A}x + \text{B}y + \text{C}z = 1. \tag{47}$$

Suppose that each of these constants is a function of a variable parameter a, viz. let $\text{A} = f(a)$, $\text{B} = \phi(a)$, $\text{C} = \psi(a)$; then, as a continuously varies, the plane will have different positions, any two consecutive ones of which will intersect in a straight line; and will, as its position varies, generate a developable surface, of which the straight line of intersection of two consecutive planes will be the generator; thus the equation to one of the planes will be

$$xf(a) + y\phi(a) + z\psi(a) = 1. \tag{48}$$

Differentiating which in reference to a, and we have

$$x f'(a) + y \phi'(a) + z \psi'(a) = 0, \tag{49}$$

which is the equation to another plane; and the line of intersection of the two is the generator of the developable surface; and from these two eliminating a, we shall obtain an equation in terms of x, y, z which will be that to the required surface: which however cannot be determined in the general case so long as the functions involved in (48) and (49) are undetermined.

308.] To find the Differential Equation to Developable Surfaces.

In (48) and (49) a is a function of x, y, z. Taking then the partial differentials of (48), we have

$$\{x f'(a) + y \phi'(a) + z \psi'(a)\} \left(\frac{da}{dx}\right) + f(a) + \psi(a) \left(\frac{dz}{dx}\right) = 0,$$

$$\{x f'(a) + y \phi'(a) + z \psi'(a)\} \left(\frac{da}{dy}\right) + \phi(a) + \psi(a) \left(\frac{dz}{dy}\right) = 0;$$

whence, by means of (49), we have

$$\left.\begin{aligned} f(a) + \psi(a) \left(\frac{dz}{dx}\right) &= 0, \\ \phi(a) + \psi(a) \left(\frac{dz}{dy}\right) &= 0, \end{aligned}\right\} \tag{50}$$

From which last two equations, eliminating (a), we have a relation of the form

$$\left(\frac{dz}{dx}\right) = \mathrm{F}\left\{\left(\frac{dz}{dy}\right)\right\};$$

whence, eliminating F, there results

$$\left(\frac{d^2 z}{dx^2}\right)\left(\frac{d^2 z}{dy^2}\right) - \left(\frac{d^2 z}{dx\,dy}\right)^2 = 0, \tag{51}$$

which is the general differential equation to developable surfaces.

Another differential equation equivalent to (51) may be found as follows:

Suppose the equation to the surface to be

$$\mathrm{F}(x, y, z) = 0,$$

and $\text{U}, \text{V}, \text{W}$ to be its partial derived-functions; then

$$\text{U}\,dx + \text{V}\,dy + \text{W}\,dz = 0; \tag{52}$$

and suppose the equation to the plane by the consecutive intersection of which the surface is formed to be

$$\text{A}x + \text{B}y + \text{C}z = 1, \tag{53}$$

and as this is a tangent plane (see Art. 300),

$$\text{A}\,dx + \text{B}\,dy + \text{C}\,dz = 0; \tag{54}$$

$\therefore$ by comparing (52) and (54),

$$\frac{\text{U}}{\text{A}} = \frac{\text{V}}{\text{B}} = \frac{\text{W}}{\text{C}} = \lambda. \tag{55}$$

Differentiating again (52) and (54), since the tangent plane touches the surface along the generating line,

$$\text{U}\,d^2x + \text{V}\,d^2y + \text{W}\,d^2z + d\text{U}\,dx + d\text{V}\,dy + d\text{W}\,dz = 0, \tag{56}$$

$$\text{A}\,d^2x + \text{B}\,d^2y + \text{C}\,d^2z = 0, \tag{57}$$

whence replacing $\text{U}, \text{V}, \text{W}$ from (55) in terms of $\text{A}, \text{B}, \text{C}$,

$$\lambda\,\{\text{A}\,d^2x + \text{B}\,d^2y + \text{C}\,d^2z\} + d\text{U}\,dx + d\text{V}\,dy + d\text{W}\,dz = 0,$$

$$\therefore\quad d\text{U}\,dx + d\text{V}\,dy + d\text{W}\,dz = 0, \tag{58}$$

which is in fact identical with equation (49).

Comparing which with (52), we have

$$\frac{\text{U}}{d\text{U}} = \frac{\text{V}}{d\text{V}} = \frac{\text{W}}{d\text{W}} = \frac{1}{\mu}, \tag{59}$$

Let u, v, w, u', v', w' represent the several second partial derived-functions of $\text{F}(x, y, z)$, viz.

$$\left.\begin{array}{lll} u = \left(\dfrac{d^2\text{F}}{dx^2}\right), & v = \left(\dfrac{d^2\text{F}}{dy^2}\right), & w = \left(\dfrac{d^2\text{F}}{dz^2}\right), \\ u' = \left(\dfrac{d^2\text{F}}{dy\,dz}\right), & v' = \left(\dfrac{d^2\text{F}}{dz\,dx}\right) & w' = \left(\dfrac{d^2\text{F}}{dx\,dy}\right), \end{array}\right\} \tag{60}$$

so that

$$\left.\begin{array}{l} \mu\text{U} = d\text{U} = u\,dx + w'\,dy + v'\,dz, \\ \mu\text{V} = d\text{V} = w'\,dx + v\,dy + u'\,dz, \\ \mu\text{W} = d\text{W} = v'\,dx + u'\,dy + w\,dz; \end{array}\right\} \tag{61}$$

from which, and from (52), eliminating μ, dx, dy, dz, we have

$$\begin{aligned}\mathrm{U}^2(vw-u'^2) + \mathrm{V}^2(wu-v'^2) + \mathrm{W}^2(uv-w'^2)\\ +2\mathrm{V}\mathrm{W}(v'w'-uu')+2\mathrm{W}\mathrm{U}(w'u'-vv')+2\mathrm{U}\mathrm{V}(u'v'-ww')=0;\quad(62)\end{aligned}$$

which result may easily be shewn to be identical with (51) by writing the equations in the form

$$\mathrm{F}(x, y, z) = z - f(x, y) = 0.$$

These equations are of course satisfied by the equations to the cylinder and to the cone.

309.] Since every two consecutive generating lines of a developable surface intersect, a curve is formed, after the manner of an envelope (see Chapter XIII, Section 2), by the continual intersection of all these; and this must be a curve of double curvature, otherwise all the lines would be in one plane, and the developable surface would be only a plane. This curve bears the name of Edge of Regression (*Arête de Rebroussement*), and the generator of the surface is plainly always a tangent to it. Its equation may be found as follows:

Equations (48) and (49), if a be considered constant, are, taken together, the equations to a generating line of the developable surface, and therefore, from what has just been said, of the line whose envelope has to be determined: and the equation to which may therefore be found by making a to vary. Differentiating therefore (49) with respect to a, we have

$$x f''(a) + y\phi''(a) + z\psi''(a) = 0, \qquad (63)$$

and eliminating (a) between this (48) and (49), we shall get two equations in terms of x, y, and z which are those to the edge of regression. This line, as is plain from its mode of generation, bounds the developable surface towards one side of space; and on the other side the surface continues in two sheets; and thus any plane section of it, made by a plane which does not contain one of its generating lines, is a curve with a singular point in it.

310.] Hence also we arrive at a new conception of a developable surface; it is generated by a tangent of a curve of double curvature which moves continuously along the curve. Also since the osculating plane is that which contains two consecutive tangents, it may be conceived of as formed by the con-

tinuous intersection of such osculating planes. Suppose then that the equations to a curve of double curvature are given in the forms (see Ex. 2, Art. 295,)

$$x = f(a), \qquad y = \phi(a), \qquad z = \psi(a), \tag{64}$$

a being a variable parameter, and so that

$$dx = f'(a)\,da, \qquad dy = \phi'(a)\,da, \qquad dz = \psi'(a)\,da; \tag{65}$$

then, by equations (2), Art. 287, the equations to the generating line are

$$\frac{\xi - f(a)}{f'(a)} = \frac{\eta - \phi(a)}{\phi'(a)} = \frac{\zeta - \psi(a)}{\psi'(a)}, \tag{66}$$

from which the equation to the surface will be found by the elimination of (a).

Similarly also will developable surfaces be formed by the intersection of normal planes of a curve of double curvature; for suppose the equations to the curve to be of the form (64), then the equation to the normal plane is

$$\{\xi - f(a)\}f'(a) + \{\eta - \phi(a)\}\,\phi'(a) + \{\zeta - \psi(a)\}\,\psi'(a) = 0, \tag{67}$$

an equation of the form (48), and therefore manifestly that of a developable surface. Fig. 129 indicates the mode of generation of such surfaces and edges of regression.

311.] Ex. 1. To find the Equation to the surface generated by tangents to the Helix, or (what is the same thing) formed by the continuous intersection of Osculating Planes.

By Ex. 2, Art. 295, the equation to the osculating plane is

$$\eta \cos\phi - \xi \sin\phi = \frac{\zeta}{k} - a\phi; \tag{68}$$

$\therefore$ differentiating with respect to ϕ,

$$\eta \sin\phi + \xi \cos\phi = a, \tag{69}$$

whence, squaring and adding,

$$\phi = \frac{\zeta - k(\xi^2 + \eta^2 - a^2)^{\frac{1}{2}}}{ka}, \tag{70}$$

whereby (69) becomes

$$\eta \sin \frac{\zeta - k(\xi^2 + \eta^2 - a^2)^{\frac{1}{2}}}{ka} + \xi \cos \frac{\zeta - k(\xi^2 + \eta^2 - a^2)^{\frac{1}{2}}}{ka} = a, \tag{71}$$

which is the equation to the developable helicoid, or screw-surface; the edge of regression of which is the helix itself.

Similarly will the equation to the surface formed by the intersection of consecutive normal planes to the helix be found to be

$$\eta \sin \frac{k\zeta - (\eta^2 + \xi^2 - k^4 a^2)^{\frac{1}{2}}}{k^2 a} + \xi \cos \frac{k\zeta - (\eta^2 + \xi^2 - k^4 a^2)^{\frac{1}{2}}}{k^2 a} + k^2 a = 0. \quad (72)$$

Ex. 2. If the equations to the curve of double curvature be

$$\left.\begin{aligned} x^2 + y^2 + z^2 &= k^2, \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} &= 1, \end{aligned}\right\} \quad (73)$$

the student will without difficulty find the following equation to the surface formed by the intersection of normal planes

$$\left(\frac{\xi}{\alpha}\right)^{\frac{2}{3}} \pm \left(\frac{\eta}{\beta}\right)^{\frac{2}{3}} - \left(\frac{\zeta}{\gamma}\right)^{\frac{2}{3}} = 0, \quad (74)$$

which is manifestly the equation to a cone; the ambiguity of sign in the second term depending on k^2 being greater or less than b^2.

A further enquiry into the properties of developable surfaces will be beyond the scope of the present Treatise; but I cannot refrain from recommending the reader to study the works of Monge and Dupin on these subjects: works as they are of such intrinsic merit that I cannot venture to characterize them, for it may be that my praise would only tend towards disparagement.

312.] On Skew Surfaces.

Skew surfaces (see Art. 299) are those ruled surfaces, any two consecutive generating lines of which do not intersect; in the complete equations of such generators, (9) and (10), Art. 299, there are three conditions left undetermined, and these may be that the generator shall meet three directors. It is also manifest geometrically that such conditions fix the generator; for take a point chosen arbitrarily on any one to be the vertex of a cone, from which conceive two conical surfaces to be described with the other two generators as their directors; then these cones will intersect in one or more straight lines, which will be the generators of the skew surface.

Now in developable surfaces we shewed that a tangent plane at any point of the surface not only contained a generator, but touched the surface through the whole length of the generator.

In skew surfaces however it is not so; the tangent plane contains the generator, but cuts the surface at every other point along it save at that of contact. For suppose P and P′ to be any two points at a finite distance apart on the generator of a skew surface, and Q and Q′ to be two other points respectively near to them; then the tangent plane at P contains the line PQ, and that at P′ contains P′Q′; but these tangent planes cannot be identical in position, for were they so, the line joining Q and Q′ would be in the same plane with that joining P and P′, and two consecutive straight lines would intersect; and this is inconsistent with the fact of the surface being skew. Hence the tangent plane of a skew surface cuts that surface along the length of the generator, save at the point of contact.

The equations therefore to the generator of a skew surface are the following:

Let a be a variable parameter, and let the arbitrary functions introduced into the equations be so determined that the generator may pass through three directors; accordingly we have

$$\frac{x-f(a)}{\mathrm{F}(a)} = \frac{y-\phi(a)}{\Phi(a)} = \frac{z-\psi(a)}{\Psi(a)}. \tag{75}$$

313.] The differential equation to skew surfaces may be found as follows:

Let $\mathrm{F}(x, y, z)$ be the equation to the surface; and let U, V, W be its partial derived-functions; so that

$$\mathrm{U}\,dx + \mathrm{V}\,dy + \mathrm{W}\,dz = 0. \tag{76}$$

Now take three consecutive points along the generating line of such a surface; then by what was said in the last Article, the tangent plane changes its position at those three points; U, V, W therefore, which are proportional to the direction-cosines of the normal, change as we pass along the line, but the ratios $dx : dy : dz$ remain constant, because they are the same for all points of the line. Hence we have

$$\mathrm{U}\,dx + \mathrm{V}\,dy + \mathrm{W}\,dz = 0, \tag{77}$$

$$d\mathrm{U}\,dx + d\mathrm{V}\,dy + d\mathrm{W}\,dz = 0, \tag{78}$$

$$d^2\mathrm{U}\,dx + d^2\mathrm{V}\,dy + d^2\mathrm{W}\,dz = 0; \tag{79}$$

and replacing $d\mathrm{U}$, $d\mathrm{V}$, $d\mathrm{W}$ by their values given in (61), and

d^2u, d^2v, d^2w by their similar ones, (78) and (79) become respectively a quadratic and a cubic in terms of dx, dy, dz; and from them and (77) dx, dy, dz are to be eliminated, and the resulting equation, in terms of partial differential coefficients, will be that to the skew surface.

Of this class of surfaces two kinds require special mention.

314.] *Conoidal surfaces* are those skew ones, the generating line of which always passes through and is perpendicular to some straight line, say to the axis of z, and meets some director.

Let x, y, z, x', y', z' be severally the coordinates of the generator and of the director; and let the equations to the director be

$$F_1(x', y', z') = 0, \quad F_2(x', y', z') = 0; \tag{80}$$

then the equations to the generator being

$$\frac{x}{x'} = \frac{y}{y'}, \quad z = z', \tag{81}$$

(80) become

$$F_1\left(x', \frac{y}{x}x', z\right) = 0, \quad F_2\left(x', \frac{y}{x}x', z\right) = 0; \tag{82}$$

and eliminating x' we have a result of the form

$$F\left(\frac{y}{x}, z\right) = 0, \tag{83}$$

which is the general equation to conoidal surfaces.

(83) may also be put into the explicit form

$$z = f\left(\frac{x}{y}\right). \tag{84}$$

315.] The differential equations to such surfaces may be thus found; taking the partial differentials of (84) we have

$$\left(\frac{dz}{dx}\right) = \frac{1}{y} f'\left(\frac{x}{y}\right),$$

$$\left(\frac{dz}{dy}\right) = -\frac{x}{y^2} f'\left(\frac{x}{y}\right);$$

$$\therefore \quad x\left(\frac{dz}{dx}\right) + y\left(\frac{dz}{dy}\right) = 0. \tag{85}$$

Or taking the implicit form, viz. equation (83), and writing F' for the derived-function of F,

$$\left(\frac{dF}{dx}\right) = -\frac{y}{x^2}F', \quad \left(\frac{dF}{dy}\right) = \frac{1}{x}F', \quad \left(\frac{dF}{dz}\right) = F',$$

$$\therefore \quad x\left(\frac{dF}{dx}\right) + y\left(\frac{dF}{dy}\right) + 0\left(\frac{dF}{dz}\right) = 0; \tag{86}$$

the geometrical meaning of which is, that the normal to such a surface is perpendicular to the line drawn from the point of intersection perpendicular to the axis of z.

Ex. 1. To find the Equation to the Conoid whose Director is the Helix.

$$x' = a\cos\phi, \quad y' = a\sin\phi, \quad z' = ka\phi, \tag{87}$$

$$\frac{y}{x} = \frac{y'}{x'}, \quad z = z';$$

$$\therefore \quad x' = a\cos\frac{z}{ka}, \quad y' = a\sin\frac{z}{ka},$$

$$\frac{y}{x} = \tan\frac{z}{ka},$$

$$y\cos\frac{z}{ka} - x\sin\frac{z}{ka} = 0; \tag{88}$$

which surface is called the Skew Helicoid, and is that of the under surface of spiral staircases.

Ex. 2. Let the director be a circle whose plane is parallel to that of xz at a distance c from it, and whose centre is in the axis of y; see fig. 126.

Let radius of circle $= a$, and let the generator pass through and be perpendicular to the axis of x; $OM = x$, $MN = y$, $NP = z$; $OM = x'$, $ML = y'$, $LQ = z'$.

Let $$OA = CB = CE = a,$$

$$OC = AB = c,$$

$$\therefore \quad \left.\begin{aligned} x'^2 + z'^2 &= a^2, \\ y' &= c, \end{aligned}\right\} \quad \left.\begin{aligned} x' &= x, \\ \frac{y'}{z'} &= \frac{y}{z}; \end{aligned}\right\}$$

$$\therefore \quad \left.\begin{aligned} z'^2 &= a^2 - x^2, \\ y'^2 &= c^2, \end{aligned}\right\} \quad \frac{y^2}{z^2} = \frac{c^2}{a^2 - x^2},$$

$$c^2 z^2 = y^2(a^2 - x^2); \tag{89}$$

which surface is known by the name of the Cono-Cuneus of Wallis, of which the figure contains but one-fourth, the remainder being in three other octants.

316.] To find the Equation to a surface generated by a straight line moving on two Directors and always parallel to a given plane.

It is manifest from the mode of generation, that a section of the surface made by a plane parallel to the given plane is a straight line.

Let $\text{A}x + \text{B}y + \text{C}z = a$ be the equation to the plane cutting the surface and parallel to the given plane, and therefore having A, B, C constant, and a a variable parameter; and let the equation to another plane passing through the generating straight line of the surface be

$$\text{A}_1 x + \text{B}_1 y + \text{C}_1 z = 0, \tag{90}$$

that is, conceive it to pass through the origin; then A_1, B_1, and C_1 are variable and may be considered to be functions of a, so that

$$\text{A}_1 = f(a), \quad \text{B}_1 = \phi(a), \quad \text{C}_1 = \psi(a);$$

and therefore the general equation to the surface is

$$x f(\text{A}x + \text{B}y + \text{C}z) + y\phi(\text{A}x + \text{B}y + \text{C}z) + z\psi(\text{A}x + \text{B}y + \text{C}z) = 0. \tag{91}$$

To find its differential equation: to simplify the process, suppose that the director plane is parallel to that of xy; then $\text{A} = \text{B} = 0$, and $\text{C} = 1$, and the equation becomes

$$x f(z) + y\phi(z) + z\psi(z) = 0, \tag{92}$$

which may be put into the form

$$z = x\text{F}(z) + y\Phi(z); \tag{93}$$

whence

$$\left(\frac{d^2 z}{dx^2}\right)\left(\frac{dz}{dy}\right)^2 - 2\left(\frac{d^2 z}{dx\,dy}\right)\left(\frac{dz}{dx}\right)\left(\frac{dz}{dy}\right) + \left(\frac{d^2 z}{dy^2}\right)\left(\frac{dz}{dx}\right)^2 = 0. \tag{94}$$

SECTION 2.—*On Surfaces generated by the Motion of Circles.*

317.] On Surfaces of Revolution.

DEF.—A surface of revolution is generated by a curve which revolves about a straight line called the axis, and every point of which describes a circle about the axis.

Hence, if such a surface is cut by a plane perpendicular to the axis of revolution, the section is the circumference of a circle whose centre is on the axis, and all points of which are consequently at equal distances from the axis.

Let the equations to the axis be

$$\frac{\xi-a}{l} = \frac{\eta-b}{m} = \frac{\zeta-c}{n}, \tag{95}$$

(a, b, c) being a given point through which it passes, viz. the point A in fig. 127, and l, m, n being its direction-cosines.

Let x, y, z be the current coordinates of the surface; then the equation to the plane passing through (x, y, z), and perpendicular to (95), is

$$lx + my + nz = p. \tag{96}$$

Let BAQ be the axis of revolution, OB the perpendicular from the origin on it, RP the generating curve; and suppose the equation of it to be given in the form

$$\text{AP}^2 = f(\text{BQ}), \tag{97}$$

then $$(x-a)^2 + (y-b)^2 + (z-c)^2 = f(lx + my + nz), \tag{98}$$

and which is the general equation to surfaces of revolution.

If the axis of revolution be that of z, then $a = b = c = 0$, $l = m = 0$, $n = 1$, and

$$x^2 + y^2 + z^2 = f(z);$$

or, what is equivalent,

$$z = f(x^2 + y^2). \tag{99}$$

318.] To find the Differential Equation to Surfaces of Revolution.

Eliminating f from (98) according to the method of Art. 89, we have

$$m(x-a) - l(y-b) + \{n(x-a) - l(z-c)\}\left(\frac{dz}{dy}\right)$$
$$+ \{m(z-c) - n(y-b)\}\left(\frac{dz}{dx}\right) = 0. \tag{100}$$

Or putting (98) in the form

$$F(x,y,z) = F\{(x-a)^2+(y-b)^2+(z-c)^2, lx+my+nz\} = 0, \quad (101)$$

and representing by F' the derived-function of F,

$$\left.\begin{aligned} \left(\frac{dF}{dx}\right) &= \{2(x-a)+l\}\,F', \\ \left(\frac{dF}{dy}\right) &= \{2(y-b)+m\}\,F', \\ \left(\frac{dF}{dz}\right) &= \{2(z-c)+n\}\,F', \end{aligned}\right\} \quad (102)$$

$$\{m(z-c)-n(y-b)\}\left(\frac{dF}{dx}\right)+\{n(x-a)-l(z-c)\}\left(\frac{dF}{dy}\right)$$
$$+\{l(y-b)-m(x-a)\}\left(\frac{dF}{dz}\right) = 0. \quad (103)$$

The geometrical meaning of which condition is, that the normal to the surface always meets the axis of revolution.

319.] Ex. 1. To find the Equation to a Surface described by a straight line revolving about the axis of z, which however it does not meet.

Let the equations to the revolving line be

$$\frac{x'-\alpha}{L} = \frac{y'-\beta}{M} = \frac{z'-\gamma}{N};$$

then $$x'^2+y'^2 = x^2+y^2, \quad z' = z,$$

$$x' = \alpha+\frac{L}{N}(z-\gamma), \quad y' = \beta+\frac{M}{N}(z-\gamma),$$

$$\therefore \quad \{\alpha+\frac{L}{N}(z-\gamma)\}^2+\{\beta+\frac{M}{N}(z-\gamma)\}^2 = x^2+y^2. \quad (104)$$

Ex. 2. To determine the conditions that

$$Ax^2+By^2+Cz^2+2A_1yz+2B_1zx+2C_1xy$$
$$+2A_2x+2B_2y+2C_2z+K = 0 \quad (105)$$

should express a surface of revolution.

The most general form that (91) admits of, so as to be an expression of the second degree is

$$(x-a)^2+(y-b)^2+(z-c)^2 = k^2(lx+my+nz)^2; \quad (106)$$

expanding which and equating coefficients of the same powers of the variables with those of (105), we have

$$\left.\begin{aligned} A &= 1 - k^2 l^2, \\ B &= 1 - k^2 m^2, \\ C &= 1 - k^2 n^2, \end{aligned}\right\} \qquad \left.\begin{aligned} A_1 &= -k^2 mn, \\ B_1 &= -k^2 nl, \\ C_1 &= -k^2 lm; \end{aligned}\right\}$$

$$\therefore \quad -k^2 l^2 = \frac{B_1 C_1}{A_1}, \qquad -k^2 m^2 = \frac{C_1 A_1}{B_1}, \qquad -k^2 n^2 = \frac{A_1 B_1}{C_1},$$

$$\therefore \quad A - \frac{B_1 C_1}{A_1} = B - \frac{C_1 A_1}{B_1} = C - \frac{A_1 B_1}{C_1}. \tag{107}$$

320.] On Tubular Surfaces.

DEF.—Tubular surfaces are the envelopes of spheres of constant radii, whose centres are situated in a given curve, which is called the Axis of the Tube or Canal.

The general theory of envelopes having been explained in Chapter XIII, it is unnecessary to enter on the subject at any great length, but one or two points require further elucidation.

Let $F(x, y, z, a) = 0$ be the equation to the surface, involving x, y, z its current coordinates and a a variable parameter; and therefore representing a family of surfaces as a varies, and a particular individual of it for a particular value of a. Then the equation to the envelope is found by eliminating a between

$$F = 0 \quad \text{and} \quad \frac{dF}{da} = 0, \tag{108}$$

whence will generally arise an equation in terms of x, y, z.

Now although (108) thus gives the equation to a surface, yet, if a be considered a constant in them, each when taken separately represents a surface, and when taken together they represent the line of intersection of two surfaces; to this line Monge has given the name of *characteristic*. Thus conceiving of developable surfaces as formed by the intersection of consecutive planes, since two planes intersect in a straight line, a straight line is the characteristic, and is of course the generator of the developable surface.

Further imagine that, after the characteristic has been found, the variable a varies again; we shall hereby have another

characteristic determined by those two different equations, which will in general be different in form and position, and will cut the former one; whereby an envelope will be formed of such characteristics, which will be an edge of regression, and of course a curve of double curvature. Thus we shall have three equations,

$$\left.\begin{aligned} F &= 0, \\ \frac{dF}{da} &= 0, \\ \frac{d^2F}{da^2} &= 0; \end{aligned}\right\} \qquad (109)$$

from which, eliminating a, we shall have two equations in terms of x, y, z, which will by their intersection give the edge of regression, and such as we have before met with in the case of developable surfaces. Fig. 129 will perhaps give a better notion of the formation of such a curve; but we shall return to the subject in Chapter XVIII, and discuss it at such length, in a particular case, as may clear away many difficulties.

To investigate Tubular Surfaces.

Let a = the constant radius of the sphere. And let the equations to the axis be expressed in terms of a single variable parameter α, so that the equation to a sphere may be

$$\{x-f(\alpha)\}^2 + \{y-\phi(\alpha)\}^2 + \{z-\psi(\alpha)\}^2 = a^2. \qquad (110)$$

Differentiating which with respect to α,

$$\{x-f(\alpha)\}f'(\alpha) + \{y-\phi(\alpha)\}\,\phi'(\alpha) + \{z-\psi(\alpha)\}\,\psi'(\alpha) = 0; \quad (111)$$

which, taken in combination with (110) when α is constant, represents the characteristic; and as (111) represents a plane, the characteristic is manifestly a great circle of the sphere.

Differentiating (111) again, we have

$$\{x-f(\alpha)\}f''(\alpha) + \{y-\phi(\alpha)\}\,\phi''(\alpha) + \{z-\psi(\alpha)\}\,\psi''(\alpha)$$
$$- \{(f'(\alpha))^2 + (\phi'(\alpha))^2 + (\psi'(\alpha))^2\} = 0. \quad (112)$$

By means of which, and (110) and (111) if α be eliminated, there will be two equations in terms of x, y, z, which, taken in combination, are those to the edge of regression formed by the characteristics.

321.] To find the Differential Equation to Tubular Surfaces.

Let the equation to the surface be $F(x, y, z) = 0$, of which U, V, W are the partial derived-functions; and let the equation to the generating sphere be

$$(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 = a^2; \tag{113}$$

then differentiating for the surface,

$$U\,dx + V\,dy + W\,dz = 0, \tag{114}$$

and differentiating for the sphere,

$$(x-\alpha)\,dx + (y-\beta)\,dy + (z-\gamma)\,dz = 0; \tag{115}$$

$$\therefore \quad \frac{x-\alpha}{U} = \frac{y-\beta}{V} = \frac{z-\gamma}{W} = \frac{a}{P}, \tag{116}$$

if $$P^2 = U^2 + V^2 + W^2; \tag{117}$$

hence, differentiating (116),

$$\left.\begin{aligned} P\,dx &= a\,dU - \frac{U}{P}\,dP, \\ P\,dy &= a\,dV - \frac{V}{P}\,dP, \\ P\,dz &= a\,dW - \frac{W}{P}\,dP, \end{aligned}\right\} \tag{118}$$

and using the notation of Art. 308, equation (60),

$$\left.\begin{aligned} dU &= u\,dx + w'dy + v'dz, \\ dV &= w'dx + v\,dy + u'dz, \\ dW &= v'dx + u'dy + w\,dz; \end{aligned}\right\} \tag{119}$$

therefore the group (118) becomes

$$\left.\begin{aligned} (au-P)\,dx + aw'dy + av'dz &= \frac{U}{P}\,dP, \\ aw'dx + (av-P)\,dy + au'dz &= \frac{V}{P}\,dP, \\ av'dx + au'dy + (aw-P)\,dz &= \frac{W}{P}\,dP. \end{aligned}\right\} \tag{120}$$

Whence, by cross-multiplication,

$$dx\,\{(au-\text{P})\,(av-\text{P})\,(aw-\text{P})$$
$$-a^2u'^2(au-\text{P})-a^2v'^2(av-\text{P})-a^2w'^2(aw-\text{P})+2a^3u'v'w'\}$$

$$=\frac{d\text{P}}{\text{P}}\{\text{U}\,(av-\text{P})\,(aw-\text{P})-aw'\text{V}\,(aw-\text{P})$$
$$-av'\text{W}\,(av-\text{P})+a^2u'\,(u'\text{U}+v'\text{V}+w'\text{W})-2a^2u'^2\text{U}\}, \quad (121)$$

and similarly may the values of dy and dz be found.

Multiplying through therefore by U, V, W, adding, and by means of (114),

$$\text{U}^2(av-\text{P})\,(aw-\text{P})+\text{V}^2(aw-\text{P})\,(au-\text{P})+\text{W}^2(au-\text{P})\,(av-\text{P}),$$
$$-2a\,\{u'\text{VW}(au-\text{P})+v'\text{WU}(av-\text{P})+w'\text{UV}(aw-\text{P})\}$$
$$+a^2\{(u'\text{U}+v'\text{V}+w'\text{W})^2-2(u'^2\text{U}^2+v'^2\text{V}^2+w'^2\text{W}^2)\}=0. \quad (122)$$

If $a=\infty$, this condition becomes identical with that given in equation (62) for developable surfaces.

322.] Examples of Tubular Surfaces.

Ex. 1. Let the axis of the tube be a straight line whose equations are

$$\frac{\xi}{\text{L}}=\frac{\eta}{\text{M}}=\frac{\zeta}{\text{N}},$$

and the equation to the sphere be

$$(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2=a^2;$$

and therefore

$$\frac{d\xi}{\text{L}}=\frac{d\eta}{\text{M}}=\frac{d\zeta}{\text{N}},$$

$$(x-\xi)\,d\xi+(y-\eta)\,d\eta+(z-\zeta)\,d\zeta=0;$$

$$\therefore\quad (x-\xi)\,\text{L}+(y-\eta)\,\text{M}+(z-\zeta)\,\text{N}=0,$$

$$\therefore\quad \frac{\xi}{\text{L}}=\frac{\eta}{\text{M}}=\frac{\zeta}{\text{N}}=\frac{\text{L}\xi+\text{M}\eta+\text{N}\zeta}{\text{L}^2+\text{M}^2+\text{N}^2}=\frac{\text{L}x+\text{M}y+\text{N}z}{\text{L}^2+\text{M}^2+\text{N}^2},$$

whence may ξ, η, ζ be found in terms of x, y, z; and thence, substituting in the equation to the sphere,

$$\{\text{L}(\text{M}y+\text{N}z)-x(\text{M}^2+\text{N}^2)\}^2+\{\text{M}(\text{N}z+\text{L}x)-y(\text{N}^2+\text{L}^2)\}^2$$
$$+\{\text{N}(\text{L}x+\text{M}y)-z(\text{L}^2+\text{M}^2)\}^2=a^2(\text{L}^2+\text{M}^2+\text{N}^2)^2.$$

CHAPTER XVII.

ON CURVATURE OF CURVES IN SPACE, AND ON CERTAIN KINDRED AFFECTIONS.

323.]* CERTAIN principles names and modes of estimation which were discussed in Chapter XII, as to the curvature of curves, are stated with breadth sufficient to include kindred properties of curves in space; a difference however of great importance exists between the two classes, and which it is necessary at once to bring out into greater prominence. In the former case the whole of the curve lies in one plane, and the curve is therefore called a plane curve; in the present, although every two consecutive elements, or every three consecutive points, must be in one plane, viz. the osculating plane, yet the third element, or the fourth point, may be, and generally will be, in a different plane. For this reason such curves are called non-plane curves, and from this general property arise other affections of a more complex character, and which we proceed to enquire into.

Consider a portion of a curve in space, at no point of the part of which under investigation is there a point of abrupt termination, or of discontinuity, and at which the derived-functions of the equations to the curve are not indeterminate. Now as every three consecutive points must be in one plane, and as the mode of estimating curvature as explained in Art. 231 requires only three points in the curve's plane, the principles therein investigated are immediately applicable, and we propose to apply them by a similar process, viz. by drawing in the osculating plane, which is the plane containing three consecutive

* For a most masterly exposition of the properties considered in this Chapter, and for geometrical proofs of them by the infinitesimal method, the reader is requested to consult a Memoir by M. de Saint Venant in "Trentième Cahier du Journal de l' École Royal Polytechnique."—Bachelier, Paris, 1845.

points, two consecutive normals, which will generally meet at a finite distance from the curve; the ratio of the infinitesimal angle contained between which to the element of the curve is what we have before called curvature (see Art. 231), but which, for the sake of greater distinctness, we shall now call *absolute curvature*; and in accordance with the expression we shall use the following terms: *radius of absolute curvature* is the distance from the curve of the intersection of two consecutive normals drawn in the osculating plane; *centre of absolute curvature* is the point of intersection of two such normals; *angle of curvature* is the angle contained between them; and since two such normals are perpendicular to two consecutive tangents, it is equal to the angle between *them*, and is accordingly called the *angle of contingence* (see Art. 235).

Suppose then (x, y, z), $(x + dx, y + dy, z + dz)$ to be two consecutive points on a curve, and ds the distance between them; and suppose two consecutive normals to be drawn in the osculating plane, and to contain between them the angle $d\tau$; then, if ρ = radius of absolute curvature, and $d\tau$ = angle of curvature, agreeably to Art. 235,

$$ds = \pm \rho\, d\tau, \tag{1}$$

$$\rho = \pm \frac{ds}{d\tau}. \tag{2}$$

The radius of absolute curvature, which is mathematically defined by equation (2), is therefore the distance from the curve at which two consecutive normals drawn in the osculating plane intersect, or is the ratio of the element of the curve to the angle of curvature.

324.] In a curve however of the most general nature, as we pass continuously along it, the third element will be in a plane different to that of the two preceding ones; that is, two consecutive osculating planes will be inclined to each other; or, what is the same thing, two consecutive binormals are not parallel. This then is an affection different to any of those of plane curves, and which has been called by various names*, "second curvature," "torsion," "flexure," "cambrure;" we

* On the use of these terms, consult Note I. of M. de Saint Venant's Memoir in the Journal de l' École Polytechnique.

shall call it *torsion*, and curves which are affected with it we shall call non-plane curves. In plane curves it vanishes, and is of greater or less amount according to the deviation of the curve from a plane curve; we propose to measure it according to the principles of Art. 231. If therefore ds be an element of the curve, and $d\omega$ be the angle contained between two consecutive binormals, the torsion will vary directly as $d\omega$, and inversely as ds. Let us therefore define as follows:

$$\text{torsion} = \frac{d\omega}{ds}; \tag{3}$$

and suppose R to be the radius of a circle, an arc ds of which subtends an angle $d\omega$ at the centre, then

$$\mathrm{R} = \frac{ds}{d\omega}, \tag{4}$$

and torsion $= \frac{1}{\mathrm{R}}$; calling then R the radius of torsion, $d\omega$ the angle of torsion, we have the following definition of R:

The radius of torsion is the ratio of the element of a curve to the angle of inclination of two consecutive binormals.

It is of course manifest that two consecutive binormals do not of necessity intersect; but this will appear more distinctly hereafter.

It is also to be observed, that the torsion vanishes in the case of plane curves, but that the absolute curvature vanishes only in the case of straight lines; hence we shall derive analytical conditions of lines in space being plane and being straight.

And on account of these two affections, such curves have been called "curves of double curvature."

325.] On the Radius of Absolute Curvature.

Let x, y, z be the coordinates to the point on the curve at which the radius of absolute curvature is drawn: ds an element of the curve, ξ, η, ζ the coordinates to the centre of curvature, and ρ its length; so that

$$\rho^2 = (\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2, \tag{5}$$

and the point (ξ, η, ζ) lies in the osculating plane.

To abbreviate the notation, let

$$\left.\begin{aligned} dy\,d^2z - dz\,d^2y &= \text{X}, \\ dz\,d^2x - dx\,d^2z &= \text{Y}, \\ dx\,d^2y - dy\,d^2x &= \text{Z}, \end{aligned}\right\} \tag{6}$$

$$\text{X}^2 + \text{Y}^2 + \text{Z}^2 = \text{P}^2, \tag{7}$$

so that we have by reason of equation (14), Art. 290,

$$(\xi - x)\,\text{X} + (\eta - y)\,\text{Y} + (\zeta - z)\,\text{Z} = 0. \tag{8}$$

And as the centre of absolute curvature is at the point of intersection of two consecutive normals which are in the osculating plane, it is on the line of intersection of two consecutive normal planes; whence we have

$$(\xi - x)\,dx + (\eta - y)\,dy + (\zeta - z)\,dz = 0, \tag{9}$$

$$(\xi - x)\,d^2x + (\eta - y)\,d^2y + (\zeta - z)\,d^2z = ds^2, \tag{10}$$

and therefore by cross-multiplication from (8), (9) and (10),

$$\left.\begin{aligned} \xi - x &= \frac{ds^2(\text{Y}\,dz - \text{Z}\,dy)}{\text{P}^2}, \\ \eta - y &= \frac{ds^2(\text{Z}\,dx - \text{X}\,dz)}{\text{P}^2}, \\ \zeta - z &= \frac{ds^2(\text{X}\,dy - \text{Y}\,dx)}{\text{P}^2}, \end{aligned}\right\} \tag{11}$$

$\therefore \quad \rho =$

$$\pm \frac{ds^2}{\text{P}^2}\left\{(\text{Y}\,dz - \text{Z}\,dy)^2 + (\text{Z}\,dx - \text{X}\,dz)^2 + (\text{X}\,dy - \text{Y}\,dx)^2\right\}^{\frac{1}{2}}. \tag{12}$$

Which expressions may be simplified as follows:

$$\begin{aligned} \text{Y}\,dz - \text{Z}\,dy &= dz\,(dz\,d^2x - dx\,d^2z) - dy\,(dx\,d^2y - dy\,d^2x), \\ &= (dx^2 + dy^2 + dz^2)\,d^2x - dx\,(dx\,d^2x + dy\,d^2y + dz\,d^2z), \\ &= ds^2\,d^2x - dx\,ds\,d^2s, \qquad (13) \\ &= ds^3\,d.\frac{dx}{ds}. \qquad (14) \end{aligned}$$

Similarly,

$$\text{Z}\,dx - \text{X}\,dz = ds^2\,d^2y - dy\,ds\,d^2s, \tag{15}$$

$$= ds^3\,d.\frac{dy}{ds}, \tag{16}$$

$$\text{X}\,dy - \text{Y}\,dx = ds^2\,d^2z - dz\,ds\,d^2s, \tag{17}$$

$$= ds^3\,d.\frac{dz}{ds}, \tag{18}$$

$$\text{P}^2 = \text{X}^2 + \text{Y}^2 + \text{Z}^2,$$

$$= ds^2\{(d^2x)^2 + (d^2y)^2 + (d^2z)^2 - (d^2s)^2\}. \tag{19}$$

Therefore from (13), (15), (17) and (19),

$$\frac{ds^2}{\rho} = \pm\,\{(d^2x)^2 + (d^2y)^2 + (d^2z)^2 - (d^2s)^2\}^{\frac{1}{2}}, \tag{20}$$

and from (19),
$$\rho^2 = \frac{ds^6}{\text{P}^2}; \tag{21}$$

and therefore from (14), (16), (18) and (21),

$$\left.\begin{aligned} \xi - x &= \frac{\rho^2}{ds}\,d.\frac{dx}{ds}, \\ \eta - y &= \frac{\rho^2}{ds}\,d.\frac{dy}{ds}, \\ \zeta - z &= \frac{\rho^2}{ds}\,d.\frac{dz}{ds}; \end{aligned}\right\} \tag{22}$$

whence, squaring and adding, and by means of (5),

$$\frac{1}{\rho} = \pm\,\frac{1}{ds}\left\{\left(d.\frac{dx}{ds}\right)^2 + \left(d.\frac{dy}{ds}\right)^2 + \left(d.\frac{dz}{ds}\right)^2\right\}^{\frac{1}{2}}, \tag{23}$$

(20) and (23) being the same values of $\frac{1}{\rho}$, though in different forms.

If s be equicrescent, (23) may be put in the form

$$\frac{1}{\rho^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2. \tag{24}$$

326.] Let λ, μ, ν be the direction angles of the radius of absolute curvature, then

$$\left.\begin{aligned}\cos\lambda &= \frac{\xi-x}{\rho} = \frac{\rho}{ds}\,d.\frac{dx}{ds},\\ \cos\mu &= \frac{\eta-y}{\rho} = \frac{\rho}{ds}\,d.\frac{dy}{ds},\\ \cos\nu &= \frac{\zeta-z}{\rho} = \frac{\rho}{ds}\,d.\frac{dz}{ds};\end{aligned}\right\} \qquad (25)$$

and if s be equicrescent,

$$\left.\begin{aligned}\cos\lambda &= \rho\frac{d^2x}{ds^2},\\ \cos\mu &= \rho\frac{d^2y}{ds^2},\\ \cos\nu &= \rho\frac{d^2z}{ds^2};\end{aligned}\right\} \qquad (26)$$

on comparing which results with Art. 294, it appears that the radius of absolute curvature coincides in direction with the principal normal.

327.] To determine the Angle of Curvature.

Since by equation (2), Art. 323, $d\tau = \pm\frac{ds}{\rho}$, therefore from equation (23),

$$d\tau = \left\{\left(d.\frac{dx}{ds}\right)^2 + \left(d.\frac{dy}{ds}\right)^2 + \left(d.\frac{dz}{ds}\right)^2\right\}^{\frac{1}{2}}; \qquad (27)$$

$$= ds\left\{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2\right\}^{\frac{1}{2}}, \qquad (28)$$

if s be equicrescent.

Now equation (27) is remarkable and deserves attention; $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$ are the direction-cosines of the tangent, and $d\tau$ is the infinitesimal angle between it and the consecutive tangent. This result therefore may be generalized. Let $\cos a$, $\cos\beta$, $\cos\gamma$, $\cos a + d.\cos.a$, $\cos\beta + d.\cos\beta$, $\cos\gamma + d.\cos\gamma$ be

the direction-cosines of two lines inclined to each other at an infinitesimal angle; then that infinitesimal angle is equal to

$$\{(d.\cos a)^2 + (d.\cos\beta)^2 + (d.\cos\gamma)^2\}^{\frac{1}{2}} *. \tag{29}$$

328.] We may also, as follows, immediately obtain the value of ρ, and as the process exhibits the power of the infinitesimal method in its greatest simplicity, we recommend the reader to study it attentively.

Let P, Q, R, fig. 128, be three consecutive points in the curve, all three lying in the plane of the paper which is the osculating plane. Then $PQ = ds$, $QR = ds + d^2s$; produce PQ to S, making $QS = PQ$; complete the parallelograms PQRV, VQSR, so that $QS = VR = PQ = ds$, then the angle $RQS = d\tau$; and from Q as a centre, and with QS as a radius, describe a small arc ST, which is therefore perpendicular to QR; hence $QT = QS = ds$; and therefore $RT = d^2s$, and $ST = ds \times d\tau$.

Now the projections on the coordinate axes of PQ are dx, dy, dz; and of QR, or of PV, $dx+d^2x$, $dy+d^2y$, $dz+d^2z$; therefore the projections of VQ are d^2x, d^2y, d^2z; and therefore

$$VQ^2 = (d^2x)^2 + (d^2y)^2 + (d^2z)^2;$$

but

$$\begin{aligned} VQ^2 &= SR^2, \\ &= ST^2 + TR^2, \\ &= ds^2\, d\tau^2 + (d^2s)^2; \end{aligned}$$

equating which values of VQ^2, we have

$$ds^2\, d\tau^2 = (d^2x)^2 + (d^2y)^2 + (d^2z)^2 - (d^2s)^2. \tag{30}$$

And therefore by (2),

$$\frac{1}{\rho} = \pm \frac{1}{ds^2}\left\{(d^2x)^2+(d^2y)^2+(d^2z)^2-(d^2s)^2\right\}^{\frac{1}{2}}. \tag{31}$$

Hence also we have the following value of the angle of contingence,

$$d\tau = \frac{1}{ds}\left\{(d^2x)^2 + (d^2y)^2 + (d^2z)^2 - (d^2s)^2\right\}^{\frac{1}{2}}. \tag{32}$$

* Or thus: suppose $\cos a$, &c. $\cos a + d.\cos a$, &c. to be the direction-cosines of two lines making an infinitesimal angle $d\tau$ with each other; take along the lines two distances each equal to unity, and commencing from the point of intersection of the lines. Then the projections of these distances on the coordinate axes are respectively $\cos a \ldots.$, $\cos a + d.\cos a \ldots.$; and therefore the projections of the line joining their extremities are $d.\cos a$, $d.\cos\beta$, $d.\cos\gamma$, but this line measures $d\tau$, since it subtends $d\tau$, at a distance unity; and therefore

$$(d\tau)^2 = (d.\cos a)^2 + (d.\cos\beta)^2 + (d.\cos\gamma)^2.$$

329.] Again consider fig. 128, and project the parallelogram PQRV on the plane of xy; the projections on the axis of x and y of the sides PQ and QR are severally dx, dy, and $dx+d^2x$, $dy+d^2y$; so that, as was shewn in Art. 246, the area of the projected parallelogram is equal to $dx\,d^2y-dy\,d^2x$, that is, to the quantity represented by Z in Art. 325. Similarly the projections of the parallelogram PQRV on the planes of yz and zx are X and Y; also, as in Art. 246,

$$\begin{aligned}
\text{area of PQRS} &= \text{PQ} \times \text{QR} \times \sin \text{RQS}, \\
&= ds\,(ds+d^2s) \sin d\tau, \\
&= ds^2 \times d\tau, \\
&= \frac{ds^3}{\rho}.
\end{aligned} \tag{33}$$

Hence by a property of projected areas,

$$\frac{ds^6}{\rho^2} = \text{X}^2 + \text{Y}^2 + \text{Z}^2,$$

the same as equation (21). Hence also we have the following values of the direction-cosines of the binormal, which is perpendicular to the plane in which the parallelogram PQRS lies, viz.:

$$\frac{\rho \text{X}}{ds^3}, \quad \frac{\rho \text{Y}}{ds^3}, \quad \frac{\rho \text{Z}}{ds^3}. \tag{34}$$

330.] On Torsion and Radius of Torsion.

Since in Art. 324 $d\omega$ is the angle between two consecutive binormals, and since the direction-cosines of the first binormal are

$$\frac{\text{X}}{\text{P}}, \quad \frac{\text{Y}}{\text{P}}, \quad \frac{\text{Z}}{\text{P}},$$

or in terms of (34),

$$\frac{\rho \text{X}}{ds^3}, \quad \frac{\rho \text{Y}}{ds^3}, \quad \frac{\rho \text{Z}}{ds^3};$$

it follows from equation (29), Art. 327, that

$$d\omega = \left\{ \left(d.\frac{\text{X}}{\text{P}}\right)^2 + \left(d.\frac{\text{Y}}{\text{P}}\right)^2 + \left(d.\frac{\text{Z}}{\text{P}}\right)^2 \right\}^{\frac{1}{2}}; \tag{35}$$

$$\therefore \quad d\omega^2 = \left(\frac{\text{P}d\text{X} - \text{X}d\text{P}}{\text{P}^2}\right)^2 + \left(\frac{\text{P}d\text{Y} - \text{Y}d\text{P}}{\text{P}^2}\right)^2 + \left(\frac{\text{P}dz - \text{Z}d\text{P}}{\text{P}^2}\right)^2, \tag{36}$$

$$= \frac{\mathrm{P}^2(d\mathrm{X}^2 + d\mathrm{Y}^2 + d\mathrm{Z}^2) - \mathrm{P}^2 d\mathrm{P}^2}{\mathrm{P}^4},$$

$$= \frac{(\mathrm{X}^2+\mathrm{Y}^2+\mathrm{Z}^2)(d\mathrm{X}^2+d\mathrm{Y}^2+d\mathrm{Z}^2)-(\mathrm{X}d\mathrm{X}+\mathrm{Y}d\mathrm{Y}+\mathrm{Z}d\mathrm{Z})^2}{(\mathrm{X}^2+\mathrm{Y}^2+\mathrm{Z}^2)^2},$$

$$= \frac{(\mathrm{Y}d\mathrm{Z}-\mathrm{Z}d\mathrm{Y})^2 + (\mathrm{Z}d\mathrm{X}-\mathrm{X}d\mathrm{Z})^2 + (\mathrm{X}d\mathrm{Y}-\mathrm{Y}d\mathrm{X})^2}{(\mathrm{X}^2+\mathrm{Y}^2+\mathrm{Z}^2)^2}. \tag{37}$$

But differentiating the several terms of (6), we have

$$\left.\begin{aligned} d\mathrm{X} &= dy\, d^3z - dz\, d^3y, \\ d\mathrm{Y} &= dz\, d^3x - dx\, d^3z, \\ d\mathrm{Z} &= dx\, d^3y - dy\, d^3x; \end{aligned}\right\} \tag{38}$$

$$\therefore\ \mathrm{Y}d\mathrm{Z}-\mathrm{Z}d\mathrm{Y} = \mathrm{Y}(dx\,d^3y-dy\,d^3x)-\mathrm{Z}(dz\,d^3x-dx\,d^3z),$$

$$= dx(\mathrm{X}d^3x+\mathrm{Y}d^3y+\mathrm{Z}d^3z)-d^3x(\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz):$$

but $$\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz = 0;$$

$$\therefore\ \mathrm{Y}d\mathrm{Z}-\mathrm{Z}d\mathrm{Y} = dx(\mathrm{X}d^3x + \mathrm{Y}d^3y + \mathrm{Z}d^3z); \tag{39}$$

and as similar values are true for the other terms of the numerator of (37),

$$d\omega^2 = \frac{ds^2}{\mathrm{R}^2} = \frac{ds^2(\mathrm{X}d^3x + \mathrm{Y}d^3y + \mathrm{Z}d^3z)^2}{(\mathrm{X}^2+\mathrm{Y}^2+\mathrm{Z}^2)^2};$$

$$\therefore\ \frac{1}{\mathrm{R}} = \frac{\mathrm{X}d^3x + \mathrm{Y}d^3y + \mathrm{Z}d^3z}{\mathrm{X}^2+\mathrm{Y}^2+\mathrm{Z}^2}. \tag{40}$$

And since $\mathrm{X}^2 + \mathrm{Y}^2 + \mathrm{Z}^2 = \dfrac{ds^6}{\rho^2}$,

$$\therefore\ \frac{1}{\mathrm{R}} = \frac{\rho^2}{ds^6}\{\mathrm{X}d^3x + \mathrm{Y}d^3y + \mathrm{Z}d^3z\}, \tag{41}$$

and $$d\omega = \frac{\rho^2}{ds^5}\{\mathrm{X}d^3x + \mathrm{Y}d^3y + \mathrm{Z}d^3z\}. \tag{42}$$

Also substituting the second set of the values of the direction-cosines, given in (34),

$$\frac{ds}{\mathrm{R}} = \pm\left\{\left(d.\frac{\rho\mathrm{X}}{ds^3}\right)^2 + \left(d.\frac{\rho\mathrm{Y}}{ds^3}\right)^2 + \left(d.\frac{\rho\mathrm{Z}}{ds^3}\right)^2\right\}^{\frac{1}{2}}. \tag{43}$$

331.] In reference to singular forms of the values of $d\tau$ and $d\omega$ which have been determined by equations (27) and (42), it is to be observed that

(α) If at any point of a curve $d\tau = 0$, or which is the same thing

$$d.\frac{dx}{ds} = 0, \quad d.\frac{dy}{ds} = 0, \quad d.\frac{dz}{ds} = 0, \tag{44}$$

then there is no change of direction of the tangent as we pass to a third point on the curve; that is, the curve becomes a straight line, or the curvature is suspended, and there is what is called a point of Suspended Curvature.

(β) And if $d\tau = 0$, and changes its sign, then not only does the curve at the point run into and coincide with its tangent, but it intersects it; and we have what corresponds to a point of inflexion of plane curves, and is called a point of Inflected Curvature.

(γ) And if at all points of a curve in space $d\tau = 0$, then the line is straight as is apparent at once from the three conditions of (44).

(δ) If at any point $d\omega = 0$, or what is equivalent

$$\text{x}d^3x + \text{y}d^3y + \text{z}d^3z = 0, \tag{45}$$

the osculating plane does not change position as we pass from a third to a fourth consecutive point, and the torsion at that point is suspended, or the curve is plane; such a point is conveniently called a point of Suspended Torsion.

(ϵ) And if $d\omega = 0$, and changes its sign, then the direction in which the torsion takes place is changed, and we have what is called a point of Inflected Torsion.

(ζ) And if at all points of a curve $d\omega = 0$, then the binormals are all parallel, and the curve is plane. This, it is to be observed, is the same condition as that before found in Art. 296.

332.] We proceed now to consider certain surfaces which are generated by planes and straight lines, the positions and directions of which depend on properties of curves of double curvature; and first let us consider the effects of the intersection of consecutive normal planes.

Two consecutive normal planes of course generally intersect, for did they not, the elements of the curve to which they are perpendicular would be in the same straight line, which is absurd; and they intersect in a straight line, which, in accordance with the nomenclature employed by M. de Saint Venant, and taken from Monge and Lancret, we will call the *polar line*: the polar line manifestly passing through the centre of absolute curvature, and being perpendicular to the osculating plane. Consider again a third consecutive normal plane; this will intersect the second one in a straight line, which is a consecutive polar line, and so on; and thus a developable surface will be formed by the intersection of such normal planes; and as it is also generated by the polar lines according to the properties of developable surfaces investigated in the last Chapter, it is called the Polar Surface.

Also a curve is formed by the intersection of the polar lines; for the lines of intersection of the first two and of the last two of three consecutive normal planes lie in the second plane, and, not being parallel, necessarily intersect, and must by their intersection form a curve of double curvature, which is the edge of regression of the polar surface. To this edge of regression therefore the polar lines are tangents; and as such polar lines are perpendicular to two consecutive osculating planes of the original curve, it follows that the angle of curvature of this edge of regression is equal to the corresponding angle of torsion of the primitive curve. Also, as is manifest from the mode of generation, the normal planes of the primitive curve are osculating planes of the edge of regression; and therefore the angle of torsion of the edge of regression is equal to the angle of curvature of the primitive curve. These reciprocal properties of the angles of curvature, and of torsion of the curve, and of the edge of regression of its polar surface, are due to M. Fourier. We cannot however hence conclude, that the curvatures and torsions of the two curves are reciprocally equal, because the lengths of the corresponding elements of the two curves are not always equal.

333.] Investigation of mathematical expressions connected with polar surfaces and polar lines.

Let ξ, η, ζ be the current coordinates of the normal plane,

x, y, z the coordinates to the point on the curve at which it is drawn; then the equation to the normal plane is

$$(\xi-x)\,dx + (\eta-y)\,dy + (\zeta-z)\,dz = 0, \tag{46}$$

and the equation to the consecutive normal plane is

$$(\xi-x)\,d^2x + (\eta-y)\,d^2y + (\zeta-z)\,d^2z = ds^2, \tag{47}$$

by means of which, and of the two equations to the curve, x, y, z may be eliminated, and the resulting equation in terms of ξ, η, ζ is that of the polar surface.

334.] Again, to find the Equations to the Polar Line.

Let ξ, η, ζ be the current coordinates of the line; then, as it passes through the centre of absolute curvature, and is perpendicular to the osculating plane, or is parallel to the binormal, its equations are, by means of (22),

$$\frac{\xi - x - \dfrac{\rho^2}{ds}\, d.\dfrac{dx}{ds}}{\mathrm{X}} = \frac{\eta - y - \dfrac{\rho^2}{ds}\, d.\dfrac{dy}{ds}}{\mathrm{Y}} = \frac{\zeta - z - \dfrac{\rho^2}{ds}\, d.\dfrac{dz}{ds}}{\mathrm{Z}}. \tag{48}$$

335.] Also to find the coordinates of the point of intersection of two consecutive polar lines: that is, the coordinates to a point on the edge of regression of the polar surface: we must again differentiate (47), whereby we have

$$(\xi-x)\,d^3x + (\eta-y)\,d^3y + (\zeta-z)\,d^3z = 3\,ds\,d^2s, \tag{49}$$

and by cross-multiplication between (46), (47) and (49),

$$(\xi-x)\,(\mathrm{X}\,d^3x+\mathrm{Y}\,d^3y+\mathrm{Z}\,d^3z) = 3\,ds\,d^2s\,\mathrm{X} - ds^2\,d.\mathrm{X}, \tag{50}$$

$$= -\,ds^5\,d.\frac{\mathrm{X}}{ds^3}; \tag{51}$$

and similar values are true for the symmetrical expressions. Therefore by means of (41),

$$\left.\begin{aligned} \xi-x &= -\frac{\rho^2\mathrm{R}}{ds}\,d.\frac{\mathrm{X}}{ds^3}, \\ \eta-y &= -\frac{\rho^2\mathrm{R}}{ds}\,d.\frac{\mathrm{Y}}{ds^3}, \\ \zeta-z &= -\frac{\rho^2\mathrm{R}}{ds}\,d.\frac{\mathrm{Z}}{ds^3}. \end{aligned}\right\} \tag{52}$$

By means of which and the two equations to the curve x, y, z may be eliminated, and the two resulting equations in terms of ξ, η, ζ will be those to the edge of regression of the polar surface.

336.] Let us further consider these properties in relation to a sphere which has contact with the primitive curve.

As a circle is definite when it passes through three consecutive points on a curve, and cannot in general pass through more, so a sphere is definite when it passes through four points, provided that these four points have not such a position as to give a singular form to the sphere, as for instance to make it a plane; this is manifest geometrically, inasmuch as an infinite number of spheres may be made to pass through three points, but of these that which passes through the fourth point has a definite radius; and also algebraically, because the equation to a sphere involves four arbitrary constants, and these may be expressed in terms of the coordinates of four given points. Consider then a sphere to be so placed as to pass through a given point on a curve, and to touch the tangent to the curve at the point; the sphere passes through two consecutive points on the curve, and its centre may be at any point which is equidistant from these two; it may therefore be any where in the normal plane. Suppose also that the sphere passes a third consecutive point in addition to the former two, then its centre must also be in the second consecutive normal plane, and must therefore be in the intersection of these two normal planes; that is, it must be in the polar line, but it may be at any point in that line; that line therefore may be considered as the locus of the centres of spheres which pass through the same three consecutive points on a curve. Suppose again that the sphere also passes through four consecutive points, then its centre must be in the point at which the polar line is intersected by its consecutive polar line, and is therefore at a definite point, and the radius of the sphere is of definite length. The point then at which the centre of such a sphere must be placed is on the edge of regression of the polar surface; such edge therefore may be defined to be, the locus of the centres of spheres which pass through four consecutive points on a curve of double curvature.

Therefore, ξ, η, ζ of equations (52) are the coordinates to the centre of the osculating sphere, and $\xi-x$, $\eta-y$, $\zeta-z$ are the projections on the coordinate axes of its radius. Now after some long but not difficult reductions, equations (52) assume the forms

$$\left.\begin{aligned}\xi - x &= \rho\frac{\rho}{ds}d.\frac{dx}{ds} + \frac{\text{R}\,d\rho}{ds}\frac{\rho\text{X}}{ds^3},\\ \eta - y &= \rho\frac{\rho}{ds}d.\frac{dy}{ds} + \frac{\text{R}\,d\rho}{ds}\frac{\rho\text{Y}}{ds^3},\\ \zeta - z &= \rho\frac{\rho}{ds}d.\frac{dz}{ds} + \frac{\text{R}\,d\rho}{ds}\frac{\rho\text{Z}}{ds^3};\end{aligned}\right\} \tag{53}$$

and since $\frac{\rho}{ds}d.\frac{dx}{ds}$, ... are the direction-cosines of the radius of curvature, and $\frac{\rho\text{X}}{ds^3}$, ... are the direction-cosines of the polar line, it follows that the projections on the coordinate axes of the radius of the osculating sphere is equal to the sum of the projections on the same axes of the radius of absolute curvature, and of a line equal in length to $\frac{\text{R}.d\rho}{ds}$ measured from the centre of curvature along the polar line; and therefore as the polar line is perpendicular to the radius of absolute curvature,

$$\text{the radius of the osculating sphere} = \left\{\rho^2 + \left(\frac{\text{R}.d\rho}{ds}\right)^2\right\}^{\frac{1}{2}}. \tag{54}$$

And if the radius of absolute curvature of a curve is constant for all points of the curve, so that $d\rho = 0$, then the centres of absolute curvature and of the osculating sphere are coincident; and if at a point on a curve ρ is a maximum or a minimum, the same result follows.

And by differentiating (52) or (53) we may find $d\xi$, $d\eta$, $d\zeta$, and thereby the length of an element of the edge of regression of the polar surface; and thence, by means of Fourier's relations, the curvature and torsion of the edge of regression in terms of the coordinates of the corresponding point of the primitive curve.

337.] The above investigations lead us immediately to an enquiry respecting those properties of curves of double curvature which are analogous to evolutes of plane curves.

Let a normal line be drawn at any point of a curve of double curvature; it will be in the normal plane at the point, and will therefore touch the polar surface.

Now conceive a second and consecutive normal plane to be drawn; it will meet the first normal line on the polar surface,

and at the point of meeting let a normal line be drawn to the curve: and conceive again a third consecutive normal plane to be drawn, and to meet the second normal line: and another normal line to be drawn to the curve, and by a method similar to the former one: and so on; then will a curve be described on the polar surface, the elements of which are elements of these successive normal lines, and which curve is such that if a perfectly flexible and inextensible string be fixed at any point of it, and of such a length as when stretched will reach to the curve; then, if it be wrapped round the polar surface and along, and tangential to, the curve thereon described, the extremity of the string will describe the original curve of double curvature. On this account the curve described on the polar surface is called the Evolute, and the original curve is called the Involute with respect to it.

Thus let x, y, z be the point on the original curve at which the normal is drawn, and let the point on the polar surface at which the normal abuts be that whose coordinates are ξ, η, ζ, and let $d\sigma$ be an element of the evolute; and let r be the distance between the two points; then, as r is to be a tangent to the evolute,

$$\frac{d\xi}{d\sigma} = \frac{x-\xi}{r}, \quad \frac{d\eta}{d\sigma} = \frac{y-\eta}{r}, \quad \frac{d\zeta}{d\sigma} = \frac{z-\zeta}{r}.$$

Also
$$r^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2,$$

$$-r\,dr = (x-\xi)\,d\xi + (y-\eta)\,d\eta + (z-\zeta)\,d\zeta,$$

$$-dr = \frac{d\xi^2 + d\eta^2 + d\zeta^2}{d\sigma} = \pm\, d\sigma;$$

$$\therefore \quad dr \pm d\sigma = 0,$$

and taking the negative sign $r - \sigma =$ a constant; so much of the string therefore is taken off from its length by the wrapping, as to leave the remainder equal to the distance of the point on the old curve from the point on the evolute where the wrapping ends.

Hence if from two points on an evolute tangents to the evolute are drawn to the involute, the difference of their lengths is equal to the length of the arc of the evolute between the points of contact.

As the basis of the construction of the evolute thus far has been a normal line at a given point of the original curve chosen arbitrarily, so may any other normal line be taken, and as near to the other as we please; and thus there may be any number of evolutes, all of which will be on the polar surface, and which may therefore be considered as the locus surface of such evolutes.

The locus of the centres of absolute curvature is not an evolute, although it is a curve described on the polar surface; and for this reason; suppose P, Q, R ...... to be consecutive points on the curve, and p, q, r ...... to be the centres of curvature corresponding to the points; then, if the line pqr ...... were an evolute of PQR ......, the arc pq should lie in the line Pp produced, the arc qr should lie in Qq produced, and so on: and such can be the case only when Pp, Qq, Rr ...... are two and two in the same plane; but all these lie severally in different planes, viz. in the osculating planes at P, Q, R ...... The radii of absolute curvature therefore cannot meet, and by their intersections form an evolute, unless all are in the same plane, or, in other words, unless the curve is plane. A surface however is generated by such radii of curvature, which is ruled, but of the class termed Skew.

There is also another remarkable property of evolutes of curves of double curvature, viz. when the polar surface is developed into a plane, they become straight lines all diverging from the same point. For consider two consecutive normal planes Pp and Qq drawn at P and Q, and imagine the second one to turn about the polar line in Pp, until the two planes coincide; then, as is manifest from the construction, Q falls on and coincides with P, and the line Pp coincides with Qq, so that pq, the element of the evolute, coincides in direction with Pp; and as a similar result would follow from a similar operation being performed on the other normal planes, it follows that the evolute of which pq is an element becomes a straight line emanating from P, the point into which the whole original curve becomes absorbed; and as a similar result is true of all the other evolutes, it follows that when the polar surface is developed, the evolutes become a pencil of straight lines diverging from the point into which the curve falls.

And as the length of an element of an evolute is not altered

by the development, it follows that the element of an evolute is the shortest distance between the two extremities of the element. The evolute therefore is the geodesic line between any two points on the polar surface.

338.] To enable the reader to obtain an adequate conception of the results of the last few Articles, the geometrical figure 129* is given.

Let $P_1, P_2, P_3, P_4 \ldots\ldots$ be successive points in a curve of double curvature; and through the middle points $M_1, M_2, M_3 \ldots\ldots$ of successive elements let the normal planes $L_1, L_2, L_3 \ldots\ldots$ be drawn, intersecting each other consecutively in the straight lines $A_1 B_1, A_2 B_2 \ldots\ldots$ which are therefore the polar lines; and the surface formed by the intersection of the normal planes is the developable polar surface; then, as all the elements of the curve are not in the same plane, the polar lines are not parallel, and therefore intersect consecutively, and thereby form an envelope, viz. the non-plane curve $Q_1 Q_2 Q_3 \ldots\ldots$, which is the edge of regression of the polar surface.

Also let the normal planes L_1, L_2 be cut by the osculating plane containing the elements $P_1 P_2$ and $P_2 P_3$, and which is therefore perpendicular to L_1 and L_2, and let the lines of intersection be $M_1 C_1$, $M_2 C_1$; then C_1 is the centre of absolute curvature of the curve at P_1, and $P_1 C_1$ is the radius of absolute curvature.

Again, let the consecutive osculating plane be drawn containing the elements $P_2 P_3$ and $P_3 P_4$, and let its lines of intersection with L_2 and L_3 be $M_2 C_2$ and $M_3 C_2$; then C_2 is the centre of absolute curvature of the curve at P_2, and $P_2 C_2$ is the radius of absolute curvature. It is manifest now that the line $M_2 C_2$ does not coincide with $M_2 C_1$, because they are the lines of intersection of the same plane L_2 by different planes; $M_2 C_2$ therefore does not cut $A_1 B_1$ in the point C_1; and therefore the consecutive radii $M_1 C_1$ and $M_2 C_2$ do not meet. The successive centres of curvature therefore do not arise from the intersection of consecutive radii of curvature, and consequently these radii are not tangents to the locus of the centres; and therefore it follows that the curve $C_1 C_2 \ldots\ldots$ cannot be regarded as an evo-

* The figure is the same as that given by Monge in his "Application d'Analyse," and thence has been copied into most of the ordinary text books.

lute of the original curve. It is manifest however from the construction that such will be the case, if the original curve be plane.

The diagram gives us also a clear notion of the formation of evolutes. From M_1 let any line $M_1 D_1$ be drawn in the normal plane, meeting at D_1 the polar surface, to which it is tangential; and from D_1 let the line $D_1 M_2$ be drawn to M_2, the middle point of the next element; then this line lies in the consecutive normal plane, and is tangential to the polar surface, and has an element $D_1 D_2$ in contact with it; and let a similar process be continued on other consecutive normal planes; then there will be described on the polar surface a curve $D_1 D_2$......, such that each successive element on it being produced will pass through and be normal to the original curve, and such that the difference between two successive lines drawn from the new to the old curve is equal to an element of the new curve, as is manifest from the construction; therefore the curve $D_1 D_2$...... is an evolute to the original curve, which is called an involute relatively to it. Hereby also a developable surface will be formed of which the curve $D_1 D_2$...... is the edge of regression, the osculating planes to which are those containing the successive elements of the original curve.

And as the first normal $P_1 D_1$ was drawn to a point D_1 chosen arbitrarily, so might any other point have been taken on the polar line $B_1 A_1 Q_1$; and thus there may be any number of evolutes on the polar surface, and the polar surface may be considered as the locus-surface of such curves.

It is manifest from the geometry that if the polar surface be developed into a plane, $D_1 D_2$...... would become a straight line; therefore $D_1 D_2$...... is a geodesic line on the polar surface.

339.] From the figure also may be deduced results, some of which are identical with those above investigated by algebraical methods.

Through the points M_1, M_2......, the middle points of successive elements, let there be drawn the binormals $M_1 L_1$, $M_2 L_2$...... which are parallel to the polar lines $A_1 B_1$, $A_2 B_2$......; then the angle $A_1 Q_1 A_2$, being that between two successive polar lines, is equal to the angle of torsion. Also let the line $M_2 C_2$ cut the polar line $A_1 B_1$ in I; then we have the following values:

$$\text{M}_1\text{C}_1 = \rho = \text{radius of absolute curvature},$$
$$\text{M}_2\text{I} = d\rho = \text{differential of ditto},$$
$$\text{M}'_1\text{Q}_1 = \text{radius of osculating sphere},$$
$$\text{A}_1\text{Q}_1\text{A}_2 = \text{angle of torsion} = d\omega = \frac{ds}{\text{R}}.$$

Since then C_2I may be considered as an arc of a circle, subtending an angle $\text{A}_1\text{Q}_1\text{A}_2$ at Q_1, we have

$$\text{C}_2\text{I} = \text{Q}_1\text{C}_2 \times \text{angle } \text{A}_1\text{Q}_1\text{A}_2;$$
$$\therefore \quad d\rho = \text{Q}_1\text{C}_2 \times d\omega,$$
$$\therefore \quad \text{Q}_1\text{C}_2 = \frac{d\rho}{d\omega} = \text{R}\frac{d\rho}{ds}; \tag{55}$$

and since $\text{M}_1\text{C}_1\text{Q}_1$ is a right angle,

$$\text{M}_1\text{Q}_1{}^2 = \text{M}_1\text{C}_1{}^2 + \text{C}_1\text{Q}_1{}^2,$$
$$\text{radius of osculating sphere} = \left\{\rho^2 + \left(\text{R}\frac{d\rho}{ds}\right)^2\right\}^{\frac{1}{2}}, \tag{56}$$
$$\tan \text{Q}_1\text{M}_1\text{C}_1 = \frac{\text{Q}_1\text{C}_1}{\text{P}_1\text{C}_1} = \frac{\text{R}\,d\rho}{\rho\,ds}. \tag{57}$$

Also as M_2C_1, M_2C_2 are the lines of intersection of the normal plane at P_2, by two consecutive osculating planes which are both perpendicular to it, the angle $\text{C}_1\text{M}_2\text{C}_2$ is that between the osculating planes, and is therefore equal to the angle of torsion: that is,

$$\text{C}_1\text{M}_2\text{C}_2 = d\omega;$$

and as M_2C_1 is perpendicular to A_1B_1, C_1I may be considered the arc of a circle subtending at M_2 an angle $d\omega$; therefore

$$\text{C}_1\text{I} = \text{M}_2\text{C}_1 \times \text{angle } \text{C}_1\text{M}_2\text{C}_2,$$
$$= \rho\, d\omega; \tag{58}$$

and therefore if C_1C_2 be joined, C_1C_2 is an element of the curve-locus of the centres of curvature;

$$\therefore \quad \text{C}_1\text{C}_2{}^2 = \text{C}_1\text{I}^2 + \text{I}\text{C}_2{}^2,$$
$$= \rho^2 d\omega^2 + d\rho^2. \tag{59}$$

If the curve be plane $d\omega = 0$, and $\text{C}_1\text{C}_2 = d\rho$, in which case C_1C_2 is an element of the evolute of the original curve; but if the original curve be non-plane, the curve $\text{C}_1\text{C}_2 \ldots\ldots$ is not an evolute.

As $c_1 c_2$ is an element of the curve locus of the centres of absolute curvature, it is equal to $\{d\xi^2 + d\eta^2 + d\zeta^2\}^{\frac{1}{2}}$ of the expressions given in equations (11) or (22) of the present Chapter; that is

$$d\rho^2 + \rho^2 d\omega^2 = \left\{dx + d.\left(\frac{\rho^2}{ds}d.\frac{dx}{ds}\right)\right\}^2 + \dots\dots + \dots \quad (60)$$

whence, after several reductions,

$$\frac{ds^2}{\rho^2} + \frac{ds^2}{\mathrm{R}^2} = \left\{d.\left(\frac{\rho}{ds}d.\frac{dx}{ds}\right)\right\}^2$$

$$+ \left\{d.\left(\frac{\rho}{ds}d.\frac{dy}{ds}\right)\right\}^2 + \left\{d.\left(\frac{\rho}{ds}d.\frac{dz}{ds}\right)\right\}^2. \quad (61)$$

340.] Now this last equation deserves attention; for on referring to equations (25) it appears, that the direction-cosines of the radius of absolute curvature are

$$\frac{\rho}{ds}d.\frac{dx}{ds}, \quad \frac{\rho}{ds}d.\frac{dy}{ds}, \quad \frac{\rho}{ds}d.\frac{dz}{ds}; \quad (62)$$

and therefore on comparison of (61) with (29), it appears that the right-hand member of (61) is the square of the infinitesimal angle contained between two consecutive radii of absolute curvature. Therefore if $d\Omega$ be the angle between these, viz. the angle between $\mathrm{M}_1\mathrm{C}_1$ and $\mathrm{M}_2\mathrm{C}_2$ in fig. 129; then

$$d\Omega^2 = \left\{d.\left(\frac{\rho}{ds}d.\frac{dx}{ds}\right)\right\}^2$$

$$+ \left\{d.\left(\frac{\rho}{ds}d.\frac{dy}{ds}\right)\right\}^2 + \left\{d.\left(\frac{\rho}{ds}d.\frac{dz}{ds}\right)\right\}^2, \quad (63)$$

$$= \frac{ds^2}{\rho^2} + \frac{ds^2}{\mathrm{R}^2}, \quad (64)$$

and assuming $\mathfrak{R}$ to be such that $ds = \mathfrak{R}\,d\Omega$, we have

$$\frac{1}{\mathfrak{R}^2} = \frac{1}{\rho^2} + \frac{1}{\mathrm{R}^2}; \quad (65)$$

and calling $\frac{1}{\mathfrak{R}}$ complex flexure and $\mathfrak{R}$ the radius of complex flexure, we have

$$(\text{complex flexure})^2 = (\text{curvature})^2 + (\text{torsion})^2. \quad (66)$$

Fig. 129 affords an easy geometrical proof of (64).

341.] On the Osculating Surface of a Non-Plane Curve.

The osculating planes of a non-plane curve by their consecutive intersections form a developable surface, whose edge of regression is the curve itself; or the surface may be considered to be generated by the tangent lines in their successive positions. The tangent line therefore is the characteristic of the surface, and which is called the *osculating surface*.

Let the equation to an osculating plane be

$$(\xi-x)\mathrm{X} + (\eta-y)\mathrm{Y} + (\zeta-z)\mathrm{Z} = 0; \tag{67}$$

then the equation to the consecutive plane is

$$(\xi-x)\,d\mathrm{X} + (\eta-y)\,d\mathrm{Y} + (\zeta-z)\,d\mathrm{Z} = 0, \tag{68}$$

$$\because \quad \mathrm{X}\,dx + \mathrm{Y}\,dy + \mathrm{Z}\,dz = 0.$$

Eliminating therefore xyz from (67), (68), and the two equations to the curve, there will result an equation in terms of ξ, η, ζ which will be that to the osculating surface.

From the mode of generation of the surface, it is manifest that all involutes to the given curve lie in it; and that each involute cuts all the generating lines of the surface at right angles.

An example of a problem of this kind has been solved in Ex. 1, Art. 311.

342.] Again, suppose planes to be drawn at every point of a non-plane curve and perpendicular to the radius of absolute curvature at the point; such will by their intersections generate a developable surface, the equation to which is found as follows:

Let ξ, η, ζ be the current coordinates of the plane; then, since it is perpendicular to the radius of absolute curvature, or to the principal normal whose direction-cosines are given by equations (25), its equation is

$$(\xi-x)\,d.\frac{dx}{ds} + (\eta-y)\,d.\frac{dy}{ds} + (\zeta-z)\,d.\frac{dz}{ds} = 0, \tag{69}$$

and the equation to the consecutive plane is

$$(\xi-x)\,d^2.\frac{dx}{ds} + (\eta-y)\,d^2.\frac{dy}{ds} + (\zeta-z)\,d^2.\frac{dz}{ds} = 0, \tag{70}$$

$$\because \quad dx\,d.\frac{dx}{ds} + dy\,d.\frac{dy}{ds} + dz\,d.\frac{dz}{ds} = 0,$$

because the tangent and the principal normal are perpendicular to each other.

The elimination therefore of xyz between (69), (70), and the two equations to the curve, will leave an equation involving ξ, η, ζ which will be that to the required surface.

The planes, by the consecutive intersection of which the surface is formed, are tangential to the original curve, and are by their position perpendicular to both the osculating and normal planes to the curve at the common point. This surface therefore always cuts at right angles the osculating surface of the curve; and since to the osculating surface the curve is the edge of regression, this surface is perpendicular to the osculating surface, through the whole length of its edge of regression. Now when a developable surface is unfolded into a plane, its edge of regression becomes a straight line, as is manifest from the very form of such surfaces; therefore when the osculating surface is developed, the original curve becomes a straight line; but what effect has such a development of the osculating surface, or such a rectification of its edge of regression, on the surface which is perpendicular to it through its edge? it developes this orthogonal surface into a plane, and the line of contact of the original curve with it becomes a straight line. This surface is accordingly called *the rectifying developable surface*, as being that which touches the curve, and the curve of contact of which, with the given curve, becomes a straight line when the surface is developed into a plane. It is also for this reason that the plane which touches a curve, and is perpendicular to the radius of absolute curvature, is called *the rectifying plane**.

343.] The line of intersection of two successive rectifying planes is called the *rectifying line*; hence it is the line of intersection of the two planes whose equations are (69) and (70); therefore its equations are

$$\frac{\xi - x}{d\,.\,\frac{dy}{ds}\,d^2.\,\frac{dz}{ds} - d\,.\,\frac{dz}{ds}\,d^2.\,\frac{dy}{ds}} = \frac{\eta - y}{d\,.\,\frac{dz}{ds}\,d^2.\,\frac{dx}{ds} - d\,.\,\frac{dx}{ds}\,d^2.\,\frac{dz}{ds}}$$

$$= \frac{\zeta - z}{d\,.\,\frac{dx}{ds}\,d^2.\,\frac{dy}{ds} - d\,.\,\frac{dy}{ds}\,d^2.\,\frac{dx}{ds}}; \qquad (71)$$

also it is perpendicular to two consecutive principal normals.

* See a paper by Lancret in "Mémoires des Savans Etrangères," tome Iᵉʳᵉ, Paris, An XIV (1805).

Also by means of the denominators of the above fractions may there be determined the infinitesimal angle between two consecutive rectifying lines.

And the edge of regression of the rectifying surface may be determined by differentiating (70), and then determining two equations in terms of ξ, η, ζ between (69), (70) and its differential, and the two equations to the curve.

344.] For the sake of practice we will apply to the helix many of the results which have been arrived at in the present Chapter. As was shewn in Ex. 2, Art. 295, the equations to the curve are

$$\left.\begin{aligned} x &= a\cos\phi, \\ y &= a\sin\phi, \\ z &= ka\phi, \end{aligned}\right\} \quad \therefore \quad \left.\begin{aligned} dx &= -a\sin\phi\, d\phi = -y\,d\phi, \\ dy &= \ a\cos\phi\, d\phi \ = \quad x\,d\phi, \\ dz &= \quad ka\,d\phi \quad\ = \quad ka\,d\phi, \end{aligned}\right\} \qquad (72)$$

$$ds^2 = (1+k^2)a^2 d\phi^2;$$

and taking ϕ to be equicrescent,

$$\left.\begin{aligned} d^2x &= -x\,d\phi^2, \\ d^2y &= -y\,d\phi^2, \\ d^2z &= 0, \end{aligned}\right\} \qquad \left.\begin{aligned} d^3x &= y\,d\phi^3, \\ d^3y &= -x\,d\phi^3, \\ d^3z &= 0, \end{aligned}\right\} \qquad (73)$$

and from the results of Ex. 2, Art. 295, and from (6) and (7) of Art. 325,

$$\left.\begin{aligned} \frac{d^2x}{ds^2} &= \frac{-x}{a^2(1+k^2)}, \\ \frac{d^2y}{ds^2} &= \frac{-y}{a^2(1+k^2)}, \\ \frac{d^2z}{ds^2} &= 0, \end{aligned}\right\} \qquad \left.\begin{aligned} \mathrm{X} &= kay\,d\phi^3, \\ \mathrm{Y} &= -kax\,d\phi^3, \\ \mathrm{Z} &= a^2\,d\phi^3, \\ \mathrm{P}^2 &= a^4(1+k^2)d\phi^6; \end{aligned}\right\} \qquad (74)$$

$\therefore$ by (24) and (25) of the present Chapter,

$$\rho = a(1+k^2);$$

$$\cos\lambda = -\cos\phi, \qquad \cos\mu = -\sin\phi, \qquad \cos\nu = 0;$$

$$d\tau = \frac{d\phi}{\{1+k^2\}^{\frac{1}{2}}}. \qquad (75)$$

The radius of absolute curvature therefore coincides in direction with the radius of the base-cylinder of the helix; and as its value is constant, its extremity describes another helix of the same axis and the same thread, and which is traced on a base-cylinder whose radius is $k^2 a$.

Again, from (40) and (42),

$$\left.\begin{aligned} \text{R} &= a\frac{1+k^2}{k}, \qquad \therefore \quad \text{R} = \frac{\rho}{k}, \\ d\omega &= \frac{k}{(1+k^2)^{\frac{1}{2}}}\,d\phi. \end{aligned}\right\} \tag{76}$$

And since the radius of absolute curvature is in direction coincident with the principal normal, whose direction-cosines are given by equations (15), Art. 292, we have

$$\left.\begin{aligned} \cos l &= \frac{\text{X}}{\text{P}} = \frac{k}{(1+k^2)^{\frac{1}{2}}}\sin\phi, \\ \cos m &= \frac{\text{Y}}{\text{P}} = -\frac{k}{(1+k^2)^{\frac{1}{2}}}\cos\phi, \\ \cos n &= \frac{\text{Z}}{\text{P}} = \frac{1}{(1+k^2)^{\frac{1}{2}}}. \end{aligned}\right\} \tag{77}$$

The equation to the polar surface has been found in the last Chapter (see Ex. 1, equation (72), Art. 311), and therefore it is unnecessary to apply equations (46) and (47) to the same problem.

By means of (48) the equations to a polar line are

$$\frac{\xi + k^2 x}{ky} = \frac{\eta + k^2 y}{-kx} = \frac{\zeta - z}{a}. \tag{78}$$

Again, to find the equations to the edge of regression of the polar surface, we have from equations (52)

$$\xi = -k^2 x, \qquad \eta = -k^2 y, \qquad \zeta = z; \tag{79}$$

substituting which in the equations to the helix, viz.

$$\left.\begin{aligned} x &= a\cos\frac{z}{ka}, \\ y &= a\sin\frac{z}{ka}, \end{aligned}\right\} \tag{80}$$

we have

$$\left.\begin{aligned}\xi &= -k^2 a \cos\frac{\zeta}{ka},\\ \eta &= -k^2 a \sin\frac{\zeta}{ka};\end{aligned}\right\} \tag{81}$$

which represent a helix described on a cylinder, whose axis is the axis of the original base-cylinder, and radius $= k^2 a$, and the inclination of whose thread to the plane of xy is the same as that of the original helix.

Also from (54),

$$\text{the radius of the osculating sphere} = a(1+k^2), \tag{82}$$

and is therefore equal to the radius of absolute curvature; and similarly the coordinates to the centre of the osculating sphere are those given in (79).

From (64),

$$d\Omega = d\phi, \tag{83}$$

that is, two consecutive radii of absolute curvature are inclined to each other at the same angle as the corresponding radii of the base-cylinder.

Also from (65),

$$\mathfrak{R} = (1+k^2)^{\frac{1}{2}} a. \tag{84}$$

By equation (69) the equation to the rectifying plane is

$$\xi x + \eta y = a^2, \tag{85}$$

that is, the plane is parallel to the axis of z, and perpendicular to the radius of the base-cylinder.

And by (71) the equations to the rectifying line are,

$$\xi = x, \qquad \eta = y, \qquad \zeta = \frac{0}{0}. \tag{86}$$

CHAPTER XVIII.

ON THE CURVATURE OF SECTIONS OF CURVED SURFACES.

345.] CONSIDER on a curved surface a point which is not one of discontinuity, and at which the derived-functions of the equation to the surface are neither singular nor indeterminate. Through this point let any number of curved lines be drawn on the surface; generally the curvature of these lines will vary continuously as we pass from one to another; to the determination of their curvature we propose to apply the principles which have been elucidated in the preceding pages, and by means of them to form a conception of the actual flexure or curvature of the surface at the point under consideration.

Now our principles require an investigation into the inclination of two consecutive elements, or into the relative position of three consecutive points (see Art. 241). The first point is that under consideration, and the next two are of necessity in the same plane with it; and according as the plane of the three consecutive points does or does not contain the normal to the surface at the first point, will the process of inquiry be different.

Firstly, let the three points be such that their plane may contain the normals at the first and second points; then the two consecutive normals lie in the same plane, and therefore intersect. Accordingly the principles on which we have estimated radius of curvature are directly applicable; this is called the case of *principal normal section.*

Secondly, let the plane of the three points contain the first normal, but not the second; then the principles of estimating curvature which were adopted in the last Chapter are applicable; and we must investigate the distance from the surface at which the normal plane of the second element meets the normal to the surface at the given point, for such a point will be identical with the point of intersection on the osculating plane of two consecutive normal planes; this is the case of *ordinary normal section.*

Thirdly, the three points may be such that their plane may not contain the normal line at the given point; and we must then apply the results of Art. 325 to determine the radius of absolute curvature. This is the case of *oblique section,* the osculating plane of the two elements being inclined at a given angle to the normal of the surface.

346.] On the Curvature of the Principal Normal Sections.

Let the equation to the surface be

$$\text{F}(x, y, z) = 0, \tag{1}$$

and for the sake of abbreviating the notation let us employ the following symbols, as in Art. 308,

$$\left.\begin{array}{lll} \left(\frac{d\text{F}}{dx}\right) = \text{U}, & \left(\frac{d\text{F}}{dy}\right) = \text{V}, & \left(\frac{d\text{F}}{dz}\right) = \text{W}, \\ \left(\frac{d^2\text{F}}{dx^2}\right) = u, & \left(\frac{d^2\text{F}}{dy^2}\right) = v, & \left(\frac{d^2\text{F}}{dz^2}\right) = w, \\ \left(\frac{d^2\text{F}}{dy\,dz}\right) = u', & \left(\frac{d^2\text{F}}{dz\,dx}\right) = v', & \left(\frac{d^2\text{F}}{dx\,dy}\right) = w', \end{array}\right\} \tag{2}$$

$$\text{U}^2 + \text{V}^2 + \text{W}^2 = \text{Q}^2. \tag{3}$$

Then the equations to the normal at the point x, y, z are

$$\frac{\xi - x}{\text{U}} = \frac{\eta - y}{\text{V}} = \frac{\zeta - z}{\text{W}} = \frac{\rho}{\text{Q}} = \Omega, \tag{4}$$

where ρ is the distance between (x, y, z) and (ξ, η, ζ). Hence we have

$$\left.\begin{array}{l} \xi - x = \text{U}\Omega, \\ \eta - y = \text{V}\Omega, \\ \zeta - z = \text{W}\Omega. \end{array}\right\} \tag{5}$$

But when two consecutive normals intersect, (5) must consist with their differentials, when x y and z and therefore U V W Ω vary; accordingly, differentiating, we have

$$\left.\begin{array}{l} dx + \text{U}\,d\Omega + \Omega\,d\text{U} = 0, \\ dy + \text{V}\,d\Omega + \Omega\,d\text{V} = 0, \\ dz + \text{W}\,d\Omega + \Omega\,d\text{W} = 0, \end{array}\right\} \tag{6}$$

whence, eliminating Ω and $d\Omega$, there results

$$(\text{V}\,d\text{W} - \text{W}\,d\text{V})\,dx + (\text{W}\,d\text{U} - \text{U}\,d\text{W})\,dy + (\text{U}\,d\text{V} - \text{V}\,d\text{U})\,dz = 0; \tag{7}$$

and as this is independent of ξ, η, ζ it is true for any point on the surface, and is therefore the differential equation of lines drawn on the surface, which are such that the consecutive normals to the surface along them intersect. These lines are called the *lines of curvature*, other properties of which we shall take occasion hereafter to prove. The geometrical meaning of (7) is, that the line of intersection of the two tangent planes drawn at the extremities of ds, is at right angles to the tangent line whose direction-cosines are proportional to dx, dy, dz.

The equation to the lines of curvature may also be expressed as follows; for since

$$\text{v}\,d\text{w}-\text{w}\,d\text{v} = \text{v}(v'\,dx+u'\,dy+w\,dz)-\text{w}\,(w'\,dx+v\,dy+u'\,dz),$$
$$= (\text{v}v'-\text{w}w')\,dx+(\text{v}u'-\text{w}v)\,dy+(\text{v}w-\text{w}u')\,dz;$$

and since, similarly,

$$\text{w}\,d\text{u}-\text{u}\,d\text{w} = (\text{w}u-\text{u}v')\,dx+(\text{w}w'-\text{u}u')\,dy+(\text{w}v'-\text{u}w)\,dz,$$
$$\text{u}\,d\text{v}-\text{v}\,d\text{u} = (\text{u}w'-\text{v}u)\,dx+(\text{u}v-\text{v}w')\,dy+(\text{u}u'-\text{v}v')\,dz,$$

the equation to the lines of curvature becomes

$$(\text{v}v'-\text{w}w')\,dx^2+(\text{w}w'-\text{u}u')\,dy^2+(\text{u}u'-\text{v}v')\,dz^2$$
$$+(\text{w}v'-\text{u}w+\text{u}v-\text{v}w')\,dy\,dz+(\text{u}w'-\text{v}u+\text{v}w-\text{w}u')\,dz\,dx$$
$$+(\text{v}u'-\text{w}v+\text{w}u-\text{u}v')\,dx\,dy = 0. \quad (8)$$

And as this is of two dimensions in terms of dx, dy, dz, there are in general two distinct lines of curvature passing through every point on a surface; these are subsequently shewn to be perpendicular to each other at the point.

347.] Let therefore ρ be the length of the radius of curvature of a principal normal section; and let ξ, η, ζ be the coordinates to its extremity, so that

$$\rho^2 = (\xi-x)^2+(\eta-y)^2+(\zeta-z)^2,$$

and as ξ, η, ζ, ρ are the same for the next consecutive normal, and therefore for the next *two* consecutive points, we have

$$(\xi-x)\,dx+(\eta-y)\,dy+(\zeta-z)\,dz = 0, \quad (9)$$

$$(\xi-x)\,d^2x+(\eta-y)\,d^2y+(\zeta-z)\,d^2z = ds^2; \quad (10)$$

and as the centre of curvature is on the normal,

$$\frac{\xi - x}{\text{U}} = \frac{\eta - y}{\text{V}} = \frac{\zeta - z}{\text{W}} = \frac{\rho}{\text{Q}} = \frac{ds^2}{\text{U}\,d^2x + \text{V}\,d^2y + \text{W}\,d^2z}; \tag{11}$$

$$\therefore\quad \rho = \frac{\text{Q}\,ds^2}{\text{U}\,d^2x + \text{V}\,d^2y + \text{W}\,d^2z}. \tag{12}$$

Now differentiating the equation to the surface twice, we have

$$\text{U}\,d^2x + \text{V}\,d^2y + \text{W}\,d^2z + u\,dx^2 + v\,dy^2 + w\,dz^2$$
$$+ 2\{u'\,dy\,dz + v'\,dz\,dx + w'\,dx\,dy\} = 0, \tag{13}$$

and employing l, m, n to represent the direction-cosines of the element of the curve whose curvature we are investigating, so that

$$\frac{dx}{ds} = l, \quad \frac{dy}{ds} = m, \quad \frac{dz}{ds} = n, \tag{14}$$

we have, neglecting the ambiguity of sign,

$$\frac{\text{Q}}{\rho} = ul^2 + vm^2 + wn^2 + 2u'mn + 2v'nl + 2w'lm; \tag{15}$$

and this expression admits of the following modification:

Suppose that at the point on the surface neither U, V, nor W vanishes, then since

$$\text{U}\,dx + \text{V}\,dy + \text{W}\,dz = 0; \tag{16}$$

$$\therefore\quad \text{U}\,l + \text{V}\,m + \text{W}\,n = 0, \tag{17}$$

$$\therefore\quad \text{V}\,m + \text{W}\,n = -\text{U}\,l;$$

$$\left.\begin{aligned} \therefore\quad 2mn &= \frac{\text{U}^2l^2 - \text{V}^2m^2 - \text{W}^2n^2}{\text{V}\text{W}}, \\ \text{similarly}\qquad 2nl &= \frac{\text{V}^2m^2 - \text{W}^2n^2 - \text{U}^2l^2}{\text{W}\text{U}}, \\ 2lm &= \frac{\text{W}^2n^2 - \text{U}^2l^2 - \text{V}^2m^2}{\text{U}\text{V}}; \end{aligned}\right\} \tag{18}$$

and substituting as follows,

$$\left.\begin{aligned} \text{H} &= u + \frac{\text{U}}{\text{V}\text{W}}(\text{U}u' - \text{V}v' - \text{W}w'), \\ \text{K} &= v + \frac{\text{V}}{\text{W}\text{U}}(\text{V}v' - \text{W}w' - \text{U}u'), \\ \text{L} &= w + \frac{\text{W}}{\text{U}\text{V}}(\text{W}w' - \text{U}u' - \text{V}v'), \end{aligned}\right\} \tag{19}$$

we have

$$\frac{\text{Q}}{\rho} = \text{H}l^2 + \text{K}m^2 + \text{L}n^2. \tag{20}$$

But as the section whose curvature we are considering has an element coincident with that of a line of curvature, we must introduce the condition, and modify (20) accordingly; returning to (7) and (8),

$$\mathrm{V}\,d\mathrm{W}-\mathrm{W}\,d\mathrm{V} = (\mathrm{V}v'-\mathrm{W}w')\,dx+(\mathrm{V}u'-\mathrm{W}v)\,dy+(\mathrm{V}w-\mathrm{W}u')\,dz;$$

whence, by means of (16) and (19),

$$\mathrm{V}\,d\mathrm{W}-\mathrm{W}\,d\mathrm{V} = \mathrm{VL}\,dz-\mathrm{WK}\,dy. \tag{21}$$

Similarly,
$$\mathrm{W}\,d\mathrm{U}-\mathrm{U}\,d\mathrm{W} = \mathrm{WH}\,dx-\mathrm{UL}\,dz, \tag{22}$$

$$\mathrm{U}\,d\mathrm{V}-\mathrm{V}\,d\mathrm{U} = \mathrm{UK}\,dy-\mathrm{VH}\,dx, \tag{23}$$

whereby the differential equation to the lines of curvature becomes

$$\mathrm{U}(\mathrm{K}-\mathrm{L})\,dy\,dz+\mathrm{V}(\mathrm{L}-\mathrm{H})\,dz\,dx+\mathrm{W}(\mathrm{H}-\mathrm{K})\,dx\,dy = 0, \tag{24}$$

or
$$\mathrm{U}(\mathrm{K}-\mathrm{L})\,mn+\mathrm{V}(\mathrm{L}-\mathrm{H})\,nl+\mathrm{W}(\mathrm{H}-\mathrm{K})\,lm = 0. \tag{25}$$

Also from (17), $\mathrm{U}l+\mathrm{V}m+\mathrm{W}n = 0;$

whence, eliminating U, V, W in order, and simplifying the result by means of (20), we have

$$\frac{\mathrm{U}}{l\left(\mathrm{H}-\frac{\mathrm{Q}}{\rho}\right)} = \frac{\mathrm{V}}{m\left(\mathrm{K}-\frac{\mathrm{Q}}{\rho}\right)} = \frac{\mathrm{W}}{n\left(\mathrm{L}-\frac{\mathrm{Q}}{\rho}\right)}; \tag{26}$$

whence,
$$\frac{\mathrm{U}^2}{\mathrm{H}-\frac{\mathrm{Q}}{\rho}}+\frac{\mathrm{V}^2}{\mathrm{K}-\frac{\mathrm{Q}}{\rho}}+\frac{\mathrm{W}^2}{\mathrm{L}-\frac{\mathrm{Q}}{\rho}} = 0, \tag{27}$$

a quadratic equation, which gives the two values of ρ corresponding to the two radii of curvature of the principal normal sections.

If however at the point of the surface under investigation either U, V or W vanishes, the process, by which H, K, L have been formed, fails; and neither can the equation of the principal radii of curvature be expressed in the forms (20) and (27), nor the equation of the lines of curvature in the form (25). In this case then we are obliged to recur to the original forms, which are also most general, viz. (8) and (15).

348.] Equations* (26) give us also values for l, m, n which are the direction-cosines of the lines of curvature, and therefore of the principal normal sections; whereby it may be shewn, that the principal normal sections are at right angles to each other.

For let $l_1 m_1 n_1$, $l_2 m_2 n_2$ be the direction-cosines of the normal sections corresponding to which the radii of curvature are ρ_1, ρ_2; then we have, writing in (27) each value of ρ, and subtracting one from the other,

$$\left(\frac{Q}{\rho_1}-\frac{Q}{\rho_2}\right)\left\{\frac{U^2}{\left(H-\frac{Q}{\rho_1}\right)\left(H-\frac{Q}{\rho_2}\right)}\right.$$

$$\left.+\frac{V^2}{\left(K-\frac{Q}{\rho_1}\right)\left(K-\frac{Q}{\rho_2}\right)}+\frac{W^2}{\left(L-\frac{Q}{\rho_1}\right)\left(L-\frac{Q}{\rho_2}\right)}\right\}=0. \quad (28)$$

Hence, if ρ_1 and ρ_2 be unequal, as they will usually be, the second factor must be equal to 0; and therefore by reason of (26),

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0, \quad (29)$$

and therefore the principal normal sections intersect each other at right angles.

Hence also the lines of curvature at any point of a surface cut each other at right angles.

The radii of curvature of the principal normal sections are called the *principal radii of curvature*.

349.] To determine the Radius of Curvature of any Normal Section.

Observing the mode of determining the radius of curvature of such a section which was mentioned in Art. 345, it appears that we have to find the point of intersection of the normal whose equations are

$$\frac{\xi-x}{U}=\frac{\eta-y}{V}=\frac{\zeta-z}{W}, \quad (30)$$

and the plane which is perpendicular to the second element, and whose equation is given in (10); whence, using the same

* For this Article, as well as for a great part of the preceding one, and for Art. 350, I am indebted to Gregory's Solid Geometry, Cambridge, 1845.

notation and the same method of investigation as that of Art. 347, we have

$$\frac{Q}{\rho} = H l^2 + K m^2 + L n^2, \tag{31}$$

H, K, L being given in equations (19), and l, m, n being the direction-cosines of the element of the curve at the point on the surface, and whose curvature we are examining.

350.] Of all Radii of Curvature of Normal Sections, to determine whether any, and what, are Maxima and Minima.

For all normal sections passing through a given point Q, U, V, W, H, K, L being, or involving, the partial derived-functions of the equation to the surface at the point, are constant, but l, m, n vary. Hence we have to determine the maximum and minimum of (31), having given the conditions

$$U l + V m + W n = 0, \tag{32}$$

$$l^2 + m^2 + n^2 = 1; \tag{33}$$

$$\therefore \quad D\left(\frac{Q}{\rho}\right) = 2\{H l\,dl + K m\,dm + L n\,dn\} = 0, \quad U dl + V dm + W dn = 0, \quad l dl + m dm + n dn = 0, \tag{34}$$

and using indeterminate multipliers $-\lambda$ and $-\mu$ according to Art. 142,

$$\left.\begin{aligned} H l - \lambda U - \mu l &= 0, \\ K m - \lambda V - \mu m &= 0, \\ L n - \lambda W - \mu n &= 0, \end{aligned}\right\} \tag{35}$$

whence multiplying severally through by l, m, n, and adding

$$\frac{Q}{\rho} - \mu = 0,$$

$$\therefore \quad l = \frac{\lambda U}{H - \frac{Q}{\rho}}, \quad m = \frac{\lambda V}{K - \frac{Q}{\rho}}, \quad n = \frac{\lambda W}{L - \frac{Q}{\rho}}; \tag{36}$$

and therefore

$$\frac{U^2}{H - \frac{Q}{\rho}} + \frac{V^2}{K - \frac{Q}{\rho}} + \frac{W^2}{L - \frac{Q}{\rho}} = 0; \tag{37}$$

which last results are identical with (26) and (27); whence it follows, that of all normal sections the principal ones are those whose curvature is respectively a maximum and a minimum. Hence the normal sections of greatest and least curvature are at right angles to each other.

351.] The radius of curvature of any normal section may be expressed as follows, in terms of the radii of curvature of the principal normal section.

The proposition having been discovered by Euler, is known by the name of Euler's Theorem of Normal Sections.

Let ρ_1 and ρ_2 be the principal radii of curvature, and $l_1\, m_1\, n_1$, $l_2\, m_2\, n_2$ be the direction-cosines of the elements of the lines of curvature passing through the given point on the surface, and therefore of the directions of the principal normal sections at the point; let ρ be the radius of curvature of any other normal section, and $l\, m\, n$ the direction-cosines of its elements; and let a be the angle between its plane and that of the maximum principal radius of curvature; then we have

$$\left.\begin{aligned} \mathrm{U}l + \mathrm{V}m + \mathrm{W}n &= 0, \\ \mathrm{U}l_1 + \mathrm{V}m_1 + \mathrm{W}n_1 &= 0, \\ \mathrm{U}l_2 + \mathrm{V}m_2 + \mathrm{W}n_2 &= 0, \end{aligned}\right\} \tag{38}$$

$$\left.\begin{aligned} ll_1 + mm_1 + nn_1 &= \cos a, \\ ll_2 + mm_2 + nn_2 &= \sin a, \end{aligned}\right\} \tag{39}$$

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0. \tag{40}$$

Also considering the directions of the lines of curvature, and the normal of the surface, to constitute a system of rectangular axes at the point under consideration as the origin, we have

$$\left.\begin{aligned} l &= l_1 \cos a + l_2 \sin a, \\ m &= m_1 \cos a + m_2 \sin a, \\ n &= n_1 \cos a + n_2 \sin a, \end{aligned}\right\} \tag{41}$$

the last term of the general formula for such cosines vanishing, because the element of the normal section is perpendicular to the normal to the surface.

Also from (36),

$$\left.\begin{aligned}\left(\text{H}-\frac{\text{Q}}{\rho_1}\right)l_1 &= \lambda\text{U},\\ \left(\text{K}-\frac{\text{Q}}{\rho_1}\right)m_1 &= \lambda\text{V},\\ \left(\text{L}-\frac{\text{Q}}{\rho_1}\right)n_1 &= \lambda\text{W},\end{aligned}\right\} \quad \left.\begin{aligned}\left(\text{H}-\frac{\text{Q}}{\rho_2}\right)l_2 &= \lambda\text{U},\\ \left(\text{K}-\frac{\text{Q}}{\rho_2}\right)m_2 &= \lambda\text{V},\\ \left(\text{L}-\frac{\text{Q}}{\rho_2}\right)n_2 &= \lambda\text{W},\end{aligned}\right\} \tag{42}$$

whence multiplying severally by l, m, n, and adding, and by reason of (38),

$$\left.\begin{aligned}\text{H}\,l l_1 + \text{K}\,m m_1 + \text{L}\,n n_1 &= \frac{\text{Q}}{\rho}\cos a,\\ \text{H}\,l l_2 + \text{K}\,m m_2 + \text{L}\,n n_2 &= \frac{\text{Q}}{\rho}\sin a;\end{aligned}\right\} \tag{43}$$

and multiplying the first of (43) by $\cos a$, and the second by $\sin a$, and adding and reducing by means of (41), we have

$$\text{H}\,l^2 + \text{K}\,m^2 + \text{L}\,n^2 = \frac{\text{Q}}{\rho_1}(\cos a)^2 + \frac{\text{Q}}{\rho_2}(\sin a)^2; \tag{44}$$

and therefore by (31),

$$\frac{1}{\rho} = \frac{(\cos a)^2}{\rho_1} + \frac{(\sin a)^2}{\rho_2}, \tag{45}$$

which is Euler's Theorem; and is of importance, as by it the radius of curvature of any normal section is expressed in terms of the principal radii of curvature at the point in question.

352.] Hence we have the following proposition as to the radii of curvature of any two normal sections which are perpendicular to each other.

Let ρ and ρ' be the radii of curvature of two normal sections perpendicular to each other; then

$$\frac{1}{\rho} = \frac{(\cos a)^2}{\rho_1} + \frac{(\sin a)^2}{\rho_2},$$

$$\frac{1}{\rho'} = \frac{\left\{\cos\left(a+\frac{\pi}{2}\right)\right\}^2}{\rho_1} + \frac{\left\{\sin\left(a+\frac{\pi}{2}\right)\right\}^2}{\rho_2},$$

$$= \frac{(\sin a)^2}{\rho_1} + \frac{(\cos a)^2}{\rho_2},$$

$$\therefore \quad \frac{1}{\rho} + \frac{1}{\rho'} = \frac{1}{\rho_1} + \frac{1}{\rho_2}; \tag{46}$$

that is, the sum of the curvatures of two normal sections perpendicular to each other is constant.

353.] As an example of the preceding formulæ, let us take the ellipsoid whose equation is

$$F(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1; \tag{47}$$

$$\therefore \quad U = \frac{2x}{a^2}, \qquad V = \frac{2y}{b^2}, \qquad W = \frac{2z}{c^2},$$

$$u = \frac{2}{a^2}, \qquad v = \frac{2}{b^2}, \qquad w = \frac{2}{c^2},$$

$$u' = 0, \qquad v' = 0, \qquad w' = 0,$$

$$Q = 2\left\{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right\}^{\frac{1}{2}} = \frac{2}{p},$$

if p be the perpendicular from the centre on the tangent plane (see equation (31), Art. 284). Hence equation (27) becomes

$$\frac{x^2}{a^2(p\rho - a^2)} + \frac{y^2}{b^2(p\rho - b^2)} + \frac{z^2}{c^2(p\rho - c^2)} = 0, \tag{48}$$

the roots of which quadratic equation are the greatest and least principal radii of curvature at any point on the ellipsoid.

And as the last term of the quadratic when written in an integral form is $\frac{a^2b^2c^2}{p^4}$, it follows that the product of the greatest and least radii of curvature is invariable for all points for which p is constant.

Again, putting (48) in the form

$$\frac{x^2}{a^2\left(1 - \frac{a^2}{p\rho}\right)} + \frac{y^2}{b^2\left(1 - \frac{b^2}{p\rho}\right)} + \frac{z^2}{c^2\left(1 - \frac{c^2}{p\rho}\right)} = 0,$$

and subtracting it from the equation to the ellipsoid, we have

$$\frac{x^2}{a^2 - p\rho} + \frac{y^2}{b^2 - p\rho} + \frac{z^2}{c^2 - p\rho} = 1,$$

which is the equation to a concentric and confocal surface of the second order.

Also by means of (26), the directions of the principal normal sections may be determined at any point of the ellipsoid.

354.] We proceed now to consider certain singular values of ρ_1 and ρ_2, and the nature of a point on a surface whereat the singular values exist.

In equations (11) and (12) an ambiguity of sign exists, which is introduced in extracting the root of $\text{U}^2 + \text{V}^2 + \text{W}^2$, and therefore ρ may be affected with a + or a − sign; and the same ambiguity of sign continues in (27).

As U, V, W however do not vary for different normal sections, and as ρ is an absolute length of line, it appears from (11) that the change of sign arises from $\xi - x, \eta - y, \zeta - z$; and therefore the change of sign implies, that the centres of curvature are for different normal sections situated on different sides of the surface.

With respect to (45) it is to be borne in mind, that ρ_1 and ρ_2 are both taken with the positive sign, and that Q has the same sign, viz. +, in both. From (45) therefore it follows, that if ρ_1 and ρ_2 have the same sign, ρ has always the same sign as either of them; and that therefore all normal sections have their curvature in the neighbourhood of the point turned in the same direction. The analytical condition derived from equation (27) that this should be the case, is that

$$\frac{\text{U}^2}{\text{H}} + \frac{\text{V}^2}{\text{K}} + \frac{\text{W}^2}{\text{L}} \text{ must be } > 0. \tag{49}$$

Also it is manifest that, as ρ_1 and ρ_2 are a maximum and a minimum value of the radii of curvature, ρ always lies between them.

Again, if the signs of ρ_1 and ρ_2 are different, that is, if

$$\frac{\text{U}^2}{\text{H}} + \frac{\text{V}^2}{\text{K}} + \frac{\text{W}^2}{\text{L}} \text{ is } < 0, \tag{50}$$

then the radii of curvature of some normal sections are turned in a direction contrary to that of others, and (45) becomes

$$\frac{1}{\rho} = \frac{(\cos a)^2}{\rho_1} - \frac{(\sin a)^2}{\rho_2}; \tag{51}$$

let a' be such that $$\tan a' = \left(\frac{\rho_2}{\rho_1}\right)^{\frac{1}{2}}, \tag{52}$$

then for all values of a, from $-a'$ to $+a'$, and from $\pi - a'$ to $\pi + a'$, the radii of curvature of normal sections are turned in

the same direction; and when $a = \pm a'$, and $= \pi \pm a'$, $\rho = \infty$, then the normal section becomes a straight line in its consecutive elements which abut at the point, or the curvature is suspended (see Art. 331); and for all values of a outside those limits, the curvature of the normal sections is turned in a contrary direction.

According to our hypotheses ρ_1 is the maximum and ρ_2 is the minimum radius of curvature.

In the case in which one of the principal radii of curvature is infinite, say $\rho_1 = \infty$,

$$\rho = \frac{(\sin a)^2}{\rho_2}, \tag{53}$$

the analytical condition of which derived from (27) is

$$\frac{\text{U}^2}{\text{H}} + \frac{\text{V}^2}{\text{K}} + \frac{\text{W}^2}{\text{L}} = 0, \tag{54}$$

and which, when expanded, becomes identical with that determined in Art. 308, equation (62), as the differential equation of developable surfaces. Hence the meaning of the equation is apparent; one of the principal normal sections of the surface lies along its generating line, and therefore the curvature of such a section vanishes.

Again, suppose the two values of ρ_1 and ρ_2 to be equal and of opposite signs; then the coefficient of the second term of the quadratic (27) must be equal to zero, whereby we have

$$\text{U}^2(\text{K} + \text{L}) + \text{V}^2(\text{L} + \text{H}) + \text{W}^2(\text{H} + \text{K}) = 0. \tag{55}$$

In this case, a' of (52) $= 45°$; and therefore of the surface about the point, the curvature of one-half is turned in one direction, and that of the other half in the opposite direction.

355.] Lastly, let us consider the case when the principal radii of curvature are equal and have the same sign. Here $\rho_1 = \rho_2$, and equation (45) becomes

$$\rho = \rho_1 = \rho_2; \tag{56}$$

that is, the radii of curvature of all the normal sections at such a point are equal. The point is called an *umbilic*.

And the analytical conditions of such a point on a surface may thus be found. As all the principal radii of curvature must be equal, the direction-cosines which determine their directions

must be indeterminate; and as these are the same as those which determine the lines of curvature, equations (25) or (26) must be satisfied identically and independently of any particular values of l, m, n; but this is effected if

$$\text{H} = \text{K} = \text{L}; \tag{57}$$

these two equations therefore, together with that to the surface, determine the position of an umbilic. In these however the simultaneous values of xyz must not be such as to make to vanish either U, V or W, for, if so, the process according to which H, K, L were determined in Art. 347 fails. And if the two equations (57) are equivalent to only one, then this, together with the equation to the surface, will determine a line on the surface which is the locus of such umbilical points, and has by Monge been called the *line of spherical curvature*.

Also if $\text{H} = \text{K} = \text{L}$, we have from equation (20)

$$\rho = \frac{\text{Q}}{\text{H}},$$

and all the radii of curvature are equal.

In the case however in which either U, V, or W vanishes, and thereby H, K and L are rendered indeterminate, we may proceed as follows:

Suppose U to vanish; then returning to equation (8) we have

$$(\text{V}v'-\text{W}w')\,dx^2+\text{W}w'\,dy^2-\text{V}v'\,dz^2+(\text{W}v'-\text{V}w')\,dy\,dz$$
$$+(\text{V}w-\text{V}u-\text{W}u')\,dz\,dx+(\text{V}u'+\text{W}u-\text{W}v)\,dx\,dy = 0. \tag{58}$$

And since $\text{U} = 0$, equation (16) becomes

$$\text{V}dy + \text{W}dz = 0,$$

$$\therefore \quad dz = -\frac{\text{V}}{\text{W}}\,dy; \tag{59}$$

substituting which in (58) we have

$$(\text{V}v'-\text{W}w')\left\{dx^2 - \frac{\text{V}^2+\text{W}^2}{\text{W}^2}\,dy^2\right\}$$
$$+\frac{2\text{V}\text{W}u'-\text{W}^2v-\text{V}^2w+(\text{V}^2+\text{W}^2)u}{\text{W}}\,dy\,dx = 0, \tag{60}$$

which, for an umbilic, must be satisfied independently of $\dfrac{dy}{dx}$; hence the conditions are

$$\left.\begin{aligned} \text{when } \mathrm{U} = 0, \qquad & \mathrm{V}v' - \mathrm{W}w' = 0, \\ u = {} & \frac{\mathrm{V}^2 w - 2\mathrm{V}\mathrm{W}u' + \mathrm{W}^2 v}{\mathrm{V}^2 + \mathrm{W}^2}. \end{aligned}\right\} \tag{61}$$

Similarly the conditions are

$$\left.\begin{aligned} \text{when } \mathrm{V} = 0, \qquad & \mathrm{W}w' - \mathrm{U}u' = 0, \\ v = {} & \frac{\mathrm{W}^2 u - 2\mathrm{W}\mathrm{U}v' + \mathrm{U}^2 w}{\mathrm{W}^2 + \mathrm{U}^2}; \end{aligned}\right\} \tag{62}$$

$$\left.\begin{aligned} \text{and when } \mathrm{W} = 0, \qquad & \mathrm{U}u' - \mathrm{V}v' = 0, \\ w = {} & \frac{\mathrm{U}^2 v - 2\mathrm{U}\mathrm{V}w' + \mathrm{V}^2 u}{\mathrm{U}^2 + \mathrm{V}^2}. \end{aligned}\right\} \tag{63}$$

Hence to find all the umbilics on a surface, we must first seek the number of points which satisfy the general conditions (57); and also enquire when any and what points satisfy either of the three systems (61), (62), and (63).

356.] To apply the preceding formulæ to the ellipsoid.

$$\mathrm{F}(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0;$$

whence
$$\mathrm{U} = \frac{2x}{a^2}, \quad \mathrm{V} = \frac{2y}{b^2}, \quad \mathrm{W} = \frac{2z}{c^2},$$

$$u = \frac{2}{a^2}, \quad v = \frac{2}{b^2}, \quad w = \frac{2}{c^2},$$

$$u' = v' = w' = 0.$$

It is evident therefore that H, K, L can never be equal. Also of conditions (61), (62) and (63), the first and last lead to impossible results; and as to (62), let $\mathrm{V} = 0$, $\therefore y = 0$, and

$$\frac{1}{b^2}\left\{\frac{z^2}{c^4} + \frac{x^2}{a^4}\right\} = \frac{x^2}{a^4 c^2} + \frac{z^2}{c^4 a^2},$$

$$\therefore \quad x^2 = a^2 \frac{a^2 - b^2}{a^2 - c^2}, \qquad z^2 = c^2 \frac{b^2 - c^2}{a^2 - c^2}.$$

The ellipsoid therefore has four umbilics situated symmetrically in the plane of the greatest and least principal axes; and the tangent planes at these points are parallel to the circular plane sections of the surface: a result pregnant with consequences which are investigated generally in Art. 361.

If in the ellipsoid $a = b$, so that the surface becomes an oblate spheroid, $x = 0$, $z = \pm c$, and the umbilics are the points where the axis of revolution meets the surface.

Every point of a sphere is an umbilic; and a sphere is the only surface possessing this property.

357.] To return to the consideration of lines of curvature, and of the locus of the centres of principal curvature which is closely allied to them.

Suppose that a system of surfaces whose equation is $\mathfrak{F} = c$ intersects a given surface along a line of curvature, c being the parameter by the variation of which the different individuals of the system are formed; then $\mathfrak{U}$, $\mathfrak{V}$, $\mathfrak{W}$ being the partial-derived functions of $\mathfrak{F}$, we have

$$\mathfrak{U}\,dx + \mathfrak{V}\,dy + \mathfrak{W}\,dz = 0, \tag{64}$$

but dx, dy, dz are related by the equation of the lines of curvature of $\mathrm{F} = 0$; therefore from (7) and (64),

$$\frac{\mathrm{V}\,d\mathrm{W} - \mathrm{W}\,d\mathrm{V}}{\mathfrak{U}} = \frac{\mathrm{W}\,d\mathrm{U} - \mathrm{U}\,d\mathrm{W}}{\mathfrak{V}} = \frac{\mathrm{U}\,d\mathrm{V} - \mathrm{V}\,d\mathrm{U}}{\mathfrak{W}}, \tag{65}$$

and multiplying numerators and denominators severally by U, V, W, the sum of the numerators vanishes; therefore also

$$\mathrm{U}\,\mathfrak{U} + \mathrm{V}\,\mathfrak{V} + \mathrm{W}\,\mathfrak{W} = 0, \tag{66}$$

which is the condition that the surfaces cut each other at right angles.

If therefore a system of surfaces cuts another surface along its lines of curvature, it also cuts it at right angles.

Hence also it follows, that if two systems of surfaces, whose equations are of the forms $\mathrm{F}_1 = c_1$, $\mathrm{F}_2 = c_2$, cut each other along their lines of curvature, they intersect at right angles.

Hence also may be inferred Dupin's Theorem, viz. "If there are three systems of surfaces which intersect each other at right angles, any two of them will trace on the third its lines of curvature." *

* For an analytical proof of Dupin's Theorem I must refer the student to Gregory's Examples, Differential Calculus, Chapter XIII; and for a graphic description of the lines of curvature on the ellipsoid to M. Monge, "Application d'Analyse," and to M. Leroy, "Géométrie des Trois Dimensions."

358.] On the Locus-Surface of the Centres of Principal Curvature.

Imagine at a given point on a surface the two lines of curvature to be drawn; and at every point along one of these lines of curvature imagine normal planes to be drawn touching it; these will by their intersection generate a developable surface, cutting the given surface at right angles; the normal lines will be the characteristics of this developable surface, and the curve formed by their intersection will be its edge of regression, which will also be the locus-curve of the centres of principal curvature whose section touches the line of curvature.

Similarly, if normal planes be drawn touching the other line of curvature which passes through the given point, will another developable surface be generated which cuts the former at right angles; and also will there be another edge of regression, which is the locus-curve of these second centres of principal curvature.

And as a similar process will hold true for all points of a surface, so will the series of developable surfaces arising from the first line of curvature cut at right angles each of the series arising from the second line of curvature, and thus will space be filled with developable surfaces intersecting each other at right angles; and as the edges of regression belonging to the first line of curvature continuously vary, so will they generate a surface-locus of all the corresponding centres of curvature. And similarly will another surface be generated by the other centres of curvature. We shall hereby obtain a surface of two sheets, to each of which the normal of the surface is a tangent; and any two planes drawn through the normal and touching the two sheets are at right angles to each other.

If at any point the two sheets intersect each other so that their edges of regression have a common point, the principal radii of curvature of the original surface corresponding to that point are equal, and there is an umbilic; and if the two sheets intersect in a line, the line on the original surface corresponding to it is a line of spherical curvature (see Art. 355).

The analytical process for finding the equation of this surface-locus of two sheets is obvious enough. The equations to the surface, to its lines of curvature through a given point, and to

the points of intersection of consecutive normals, or the coordinates to the centre of curvature given by equations (11), are sufficient for eliminating $x\ y\ z$, and for giving an equation in terms of $\xi\ \eta$ and ζ.

As at an umbilic the equations (8) or (24) for determining the directions of the lines of curvature give indeterminate results, we must evaluate them by differentiation according to the method of Art. 114. On examining which it will be seen that each differentiation increases by unity the power of dx, dy, dz; and therefore as we begin with a quadratic, after one differentiation we shall have a cubic; after two differentiations a biquadratic; and so on. Suppose then that the directions are determinate after one differentiation, if all the three roots are real there will be three lines of curvature passing through the point; if two roots be impossible, there will be but one line of curvature; and such is the case at the umbilics on the ellipsoid. Similarly may there be four or more lines of curvature at an umbilic; nay, an infinite number, as is the case at the pole of a surface of revolution.

The lines of curvature are generally non-plane curves, and have a contact of only the first order with the principal normal section, which they touch at the point of contact; this is manifest from the fact that the equation of the lines of curvature involves differentials of the first order only. Thus, for example, at a point on a surface of revolution, one of the lines of curvature is the plane circle made by the plane passing through the point and perpendicular to the axis of revolution, because all normals to the surface on this line meet each other on the axis; but this line of curvature does not coincide with a principal section, because the principal section contains the normal which the plane line of curvature does not, unless accidentally the normal should be perpendicular to the axis of revolution. And the other line of curvature is the "meridian" curve; that is, a section made by a plane passing through the axis of revolution, because all the normals to the surface along this line are in its plane and meet. But this is also a principal section, because the normals lie and meet in it. Of a surface of revolution therefore the lines of curvature are both plane curves; and of the principal sections one is, and the other is not, coincident with a line of curvature.

359.] If the equation to the surface be given in the explicit form
$$z = f(x, y), \tag{67}$$
we must replace as follows; and as the results assume forms from which most of the properties of the curvature of surfaces have been derived in former text books, we give them in order to exhibit the identity of the conclusions:
$$\text{F}(x, y, z) = f(x, y) - z = 0. \tag{68}$$
Let
$$\left.\begin{array}{ll} \left(\frac{dz}{dx}\right) = p, & \left(\frac{d^2z}{dx^2}\right) = r, \\ \left(\frac{dz}{dy}\right) = q, & \left(\frac{d^2z}{dx\,dy}\right) = s, \\ & \left(\frac{d^2z}{dy^2}\right) = t; \end{array}\right\} \tag{69}$$
$$\therefore \left.\begin{array}{lll} \text{U} = p, & \text{V} = q, & \text{W} = -1, \\ u = r, & v = t, & w = 0, \\ u' = 0, & v' = 0, & w' = s; \end{array}\right\} \tag{70}$$
and also since
$$dz = p\,dx + q\,dy, \tag{71}$$
therefore the equation (8) of the lines of curvature becomes
$$dx^2\{s(1+p^2)-pqr\} + dy\,dx\{t(1+p^2)-r(1+q^2)\}$$
$$-dy^2\{s(1+q^2)-pqt\} = 0. \tag{72}$$
Now this it is to be observed is a quadratic equation in terms of $dy : dx$, and is the differential equation to the projections of the lines of curvature on the plane of xy. Suppose then that the coordinate planes are so chosen that the tangent plane at the point under consideration is parallel to the plane of xy, then $p = 0$, $q = 0$, and equation (72) becomes
$$\left(\frac{dy}{dx}\right)^2 + \frac{r-t}{s}\frac{dy}{dx} - 1 = 0; \tag{73}$$
and as the product of the two roots $= -1$, it follows that the lines of curvature are perpendicular to each other.

Hence also it immediately follows, that there is only one line of curvature through an umbilic of the ellipsoid, and that it is coincident in direction with the section of the greatest and least axes.

The equation (15) of the principal radii of curvature becomes

$$\rho = \frac{(1+p^2+q^2)^{\frac{1}{2}}}{rl^2+2slm+tm^2}; \tag{74}$$

the conditions (49), 50, (54) become

$$rt - s^2 >, <, = 0. \tag{75}$$

Hence the condition of a developable surface is (see Art. 308)

$$rt - s^2 = 0. \tag{76}$$

And the condition (55), that the two principal radii of curvature should be equal and affected with opposite signs, is

$$(1+q^2)r - 2pqs + (1+p^2)t = 0. \tag{77}$$

Also the general conditions (57) for an umbilic become

$$\frac{r}{1+p^2} = \frac{s}{pq} = \frac{t}{1+q^2}, \tag{78}$$

and of the three sets of special conditions, (61) and (62) become

$$p = 0, \quad s = 0, \quad r(1+q^2) = t, \tag{79}$$

$$q = 0, \quad s = 0, \quad t(1+p^2) = r; \tag{80}$$

(63) cannot be satisfied, since w cannot vanish.

Hence also we have a geometrical interpretation of Lagrange's condition that a function of two variables, say $z = f(x, y)$, should admit of a maximum or of a minimum value.

The condition is that (see Art. 135 and 136),

$$\left(\frac{d^2z}{dx^2}\right)\left(\frac{d^2z}{dy^2}\right) - \left(\frac{d^2z}{dxdy}\right)^2,$$

or $rt - s^2$ should be positive; whence it follows, that the principal radii of curvature must be measured in the same direction. Now at a point where the first differential coefficients vanish, the tangent plane is parallel to that of xy; if then all the radii of curvature at that point are measured in the same direction there must be a maximum or a minimum value of z: but if some are turned in one direction and others in the opposite direction, that is, if $rt - s^2$ is negative, there will be a partial maximum and a partial minimum, but no total maximum or minimum; and if $rt - s^2 = 0$, then the surface is developable, and the generating line will give a series of partial maxima or minima.

360.] **Meunier's Theorem on the Curvature of Oblique Sections of a Surface.**

The two cases of principal and of ordinary normal sections having thus been investigated, it remains for us to consider the third case of Art. 345: that in which a curve is described on a surface, but the osculating plane to which at the given point is not normal to the surface.

Let ρ' be the radius of absolute curvature (see Art. 325) of such a curve, and let $\lambda\, \mu\, \nu$ be the direction-angles of its direction; then by Art. 326, equation 26,

$$\left.\begin{aligned} \cos\lambda &= \rho' \frac{d^2x}{ds^2}, \\ \cos\mu &= \rho' \frac{d^2y}{ds^2}, \\ \cos\nu &= \rho' \frac{d^2z}{ds^2}, \end{aligned}\right\} \tag{81}$$

taking s to be equicrescent; and multiplying these severally by $\frac{\mathrm{U}}{\mathrm{Q}}, \frac{\mathrm{V}}{\mathrm{Q}}, \frac{\mathrm{W}}{\mathrm{Q}}$, and adding, we have

$$\rho' \frac{\mathrm{U}\,d^2\mathrm{x} + \mathrm{V}\,d^2y + \mathrm{W}\,d^2z}{\mathrm{Q}\,ds^2} = \frac{\mathrm{U}\cos\lambda + \mathrm{V}\cos\mu + \mathrm{W}\cos\nu}{\mathrm{Q}}, \tag{82}$$

but the right-hand member of the equation is the cosine of the angle between the radius of curvature of the oblique section and the normal to the surface, $= \cos\theta$, say; and by equation (12), if ρ be the radius of curvature of the normal section at the point, and which has the same tangent, the latter factor of the left-hand member $= \frac{1}{\rho}$,

$$\therefore \quad \rho' = \rho\cos\theta. \tag{83}$$

Hence the radius of curvature of an oblique section is equal to the projection, on the osculating plane at the point, of the radius of curvature of the normal section of the surface which has the same tangent line with the oblique section.

Hence if a sphere be described having for centre and radius the centre and radius of curvature of any normal section, all the oblique sections which touch the normal section at the point on the surface have, for osculating circles at the common point, the small circles of the sphere made by their respective planes.

361.] As whatever tends to elucidate the difficulties of an obscure subject deserves attention, I do not hesitate to introduce the following process, although it proves theorems which have been discussed in the previous Articles; and it exhibits the relations existing between the curvatures of normal sections in a remarkable light, and hereby indicates the nature of a point of a surface at which the partial derived-functions are not indeterminate.

Let the point of the surface under consideration be taken as the origin, and let the tangent plane be that of xy, and therefore the normal the axis of z. Let the equation to the surface be

$$z = f(x, y). \tag{84}$$

At an infinitesimal distance dz from the origin let a plane be drawn parallel to that of xy, and cutting the surface; the curve of section we will, after M. Ch. Dupin, call the *indicatrix*, as the form of it indicates the nature of the surface at the origin; let dx, dy, dz be the coordinates to a point on this curve; and through that point and the axis of z let a normal section be drawn, making an angle a with the plane of xz, so that $\tan a = \frac{dy}{dx}$; and let ds be the arc of the normal section of the surface between the origin and the point (dx, dy, dz); then

$$ds^2 = dx^2 + dy^2 + dz^2; \tag{85}$$

and if ρ be the radius of curvature at the origin of this normal section, and which lies along the axis of z, from the geometry of the circle, we have

$$\rho = \frac{ds^2}{2dz}; \tag{86}$$

that is, the radius of curvature of a normal section varies as the square of the distance between the point and the intersection of the normal plane with the indicatrix.

Using the notation of Art. 359, and expanding according to Art. 122, we have

$$z + dz = z + p\,dx + q\,dy + \frac{1}{1.2}\{r\,dx^2 + 2s\,dx\,dy + t\,dy^2\} + \ldots \tag{87}$$

and neglecting higher powers of the infinitesimals dx and dy,

and observing that $p = 0$, $q = 0$, because the normal at the origin is perpendicular to the axes of x and y, we have

$$dz = \frac{1}{2}\{r\,dx^2 + 2s\,dx\,dy + t\,dy^2\}, \tag{88}$$

and therefore

$$\rho = \frac{ds^2}{r\,dx^2+2s\,dx\,dy+t\,dy^2}; \tag{89}$$

which equation is equivalent to (15), and gives the value of the radius of curvature of the normal section.

As ρ is generally finite, it appears from equation (86) that dz is an infinitesimal of the same order as ds^2; therefore in equation (85) dz^2 must be neglected, and we have

$$ds^2 = dx^2 + dy^2. \tag{90}$$

Hence

$$\rho = \frac{dx^2+dy^2}{r\,dx^2+2s\,dx\,dy+t\,dy^2}, \tag{91}$$

$$= \frac{1}{r(\cos a)^2+2s\sin a\,\cos a+t(\sin a)^2}; \tag{92}$$

which result is the same as equation (74), and from which therefore the properties of maxima and minima radii of curvature might be deduced.

To return to the equation of the indicatrix, viz. (88); dz, being the distance between the parallel planes of xy and of that of the curve, is constant; and dx and dy are the rectangular coordinates to the indicatrix, the origin being at the point where the axis of z cuts its plane, and ds is the radius vector; hence, replacing dx and dy by ξ and η, and dz by $\frac{c}{2}$, we have

$$r\xi^2 + 2s\xi\eta + t\eta^2 = c, \tag{93}$$

which is an equation of the second degree, referred to its centre as origin, and represents an ellipse or hyperbola according as $rt - s^2$ is positive or negative; and represents a circle if $r = t$, and $s = 0$; and two parallel straight lines if $rt - s^2 = 0$. Hence we conclude, that if a surface be cut by a plane parallel and infinitesimally near to a tangent plane, the curve of section is either an ellipse, an hyperbola, or two parallel straight lines: the ellipse of course admitting of the variety of a circle, and the hyperbola in certain cases being rectangular.

If the indicatrix be an ellipse the surface is wholly concave towards it, such as is the case at all points of an ellipsoid; and, if it be an hyperbola, some part of the surface at the point has its curvature turned in one direction and some part in the opposite; and if the indicatrix be two parallel straight lines, the surface is concave towards them in a direction perpendicular to them, but is in a straight line in a direction parallel to them.

Also since ds^2 is the central radius vector of the indicatrix, and since the radius of curvature of the normal section varies as ds^2, the latter quantity partakes of whatever *singular* values the former admits of.

In the ellipse all the radii vectores are real, therefore if the indicatrix be an ellipse, that is, if $rt - s^2$ is greater than 0, all the radii of curvature of normal sections are turned in the same direction: the radii vectores of the ellipse have two maxima and two minima values, which are at right angles to each other; therefore the radii of curvature of normal sections have values respectively a maximum and a minimum, which are perpendicular to each other. In the circle all the radii vectores are equal; therefore, if $r = t$ and $s = 0$, all the radii of curvature of normal sections are equal, and there is an umbilic; hence is manifest the reason why the tangent plane at an umbilic of an ellipsoid is parallel to the planes of circular sections.

In the hyperbola some of the radii vectores are real and some are impossible, therefore if $rt - s^2$ be less than 0, the radii of curvature of normal sections are turned in one direction for all real radii vectores of the hyperbola, and in the opposite direction for the impossible ones, the asymptotes being the lines bounding the parts which have their curvatures turned in opposite directions; and if the hyperbola be rectangular, equal portions of the surface at the point have their curvatures turned in opposite directions. Hence also, as the principal axes of the hyperbola are at right angles, one being real and the other being impossible, so will the sections of greatest and least curvature be at right angles to each other, and the radii will be turned in opposite directions.

If $rt - s^2 = 0$, that is, if the indicatrix be two parallel straight lines, the origin being at a middle point between them, the radii vectores which are perpendicular to the lines are the least, and the normal section coincident with them is that of

greatest curvature; but as the line, which is parallel to and bisects them, never meets them, the corresponding radius of curvature is infinite, and the curvature of the coincident normal section vanishes. This is manifestly the case with developable surfaces.

362.] Hence also it is plain, that if the condition of *osculation* of two surfaces be made to depend on the second derived-functions as well as the first being the same in both, or in other words, on the two surfaces having the same indicatrix, a surface of the second order can always be found to osculate to a given surface at a given point; and that, in the case of an umbilic, the surface may be a sphere, and in the developable surface it becomes a cylinder.

363.] On the measure of Curvature of a Surface at a given Point.

Gauss in his celebrated memoir "*Disquisitiones Generales circa Superficies Curvas*," has introduced a definition of curvature of a surface which is derived analogously from the means of measuring the curvature of a plane curve, and from his definition has deduced a mathematical estimate of curvature.

Suppose Δs to be the finite arc of a plane curve commencing at a point P; and at the extremities of Δs let two normals to the curve be drawn. In the same plane take a circle whose radius is unity, and through its centre let two radii be drawn parallel to the two normals at the extremities of Δs, and let the radii include an angle or arc $\Delta\psi$; then the limit $\frac{d\psi}{ds}$ towards which $\frac{\Delta\psi}{\Delta s}$ converges, when the arc of the original curve is infinitesimal, is what we call (see Art. 232) the curvature of the curve at the point P.

Imagine now upon a curved surface a finite area enclosed by a contour, within which is a given point P; and also imagine a sphere whose radius is unity; and suppose normals to the surface to be drawn at every point of the enclosing contour, and radii of the sphere to be drawn parallel to these normals; by this process a spherical area will be enclosed on the surface of the sphere.

Let ΔS be the area enclosed by the contour, and $\Delta\Sigma$ the area of the enclosed figure on the surface of the sphere; then

the ratio $\frac{d\Sigma}{dS}$ towards which $\frac{\Delta\Sigma}{\Delta S}$ converges, when the contour becomes infinitesimal but still encloses the point P, is the curvature of the surface at the point P.

Let the area ΔS on the given surface be a rectangle contained by four lines of curvature, and $\Delta\phi_1$ $\Delta\phi_2$ be the angles subtended at the centres of principal curvature by two adjacent sides of the rectangle,

$$\therefore \quad \Delta S = \rho_1 \rho_2 \Delta\phi_1 \Delta\phi_2; \tag{94}$$

and similarly

$$\Delta\Sigma = \Delta\phi_1 \Delta\phi_2, \tag{95}$$

$$\therefore \quad \text{curvature} = \frac{d\Sigma}{dS} = \frac{1}{\rho_1 \rho_2}; \tag{96}$$

and the curvature of the surface at any point is equal to the product of the curvatures of the principal normal sections at the same point.

The truth of the result is manifestly independent of the form of the small area; for whatever its form be, it can always be divided into a number of infinitesimal rectangles, for every one of which the result of equation (96) will be true within an infinitesimal; and therefore by simple addition the aggregate of all, which is expressed by equation (96), will be true also. The curvature then will be affected with a positive or a negative sign, according as the radii of the principal normal sections have the same or different signs.

In the case of developable surfaces, one of the principal normal sections has an infinite radius of curvature; it would therefore follow from (96), that the curvature of a developable surface is zero: but such is the case only with a plane. We must therefore retrace our steps and modify the process in the following manner, by operating on a right circular cylinder whose radius is unity instead of on a sphere:

Let the two containing sides of the rectangular area on the given surface, and which are coincident with the lines of curvature, be Δs_1 and Δs_2; of which let Δs_2 lie along a generating line of the developable surface, and Δs_1 be at right angles to it; then in the limit, if $d\phi_1$ be the angle subtended by ds_1 at the centre of principal curvature, $ds_1 = \rho_1 d\phi_1$. Let the axis of the cylinder be parallel to the generating line of the developable at the given point, and from a point in the axis of the

cylinder let normals be drawn to the cylinder parallel to normals drawn to the developable surface along Δs_1, and let $\Delta \sigma_1$ be the intercepted arc of the circle on the surface of the cylinder; then $\Delta \sigma_1 = \Delta \phi_1$; also let a line equal to Δs_2 be taken on the surface of the cylinder and perpendicular to $\Delta \sigma_1$ through one of its extremities; by this process therefore

$$dS = \rho_1 \, d\phi_1 \, ds_2,$$

$$d\Sigma = d\phi_1 \, ds_2,$$

$$\therefore \quad \frac{d\Sigma}{dS} = \frac{1}{\rho_1}; \tag{97}$$

and therefore the curvature of a developable surface at a given point is equal to the curvature of the principal normal section which is perpendicular to a generating line.

CHAPTER XIX.

ON THE CALCULUS OF OPERATIONS.

364.] IN this the concluding Chapter of the volume it is my intention to exhibit several results which have been arrived at in the preceding pages under a new and most general form, viz. as they are presented in the Calculus of Operations; and also to prove some few of its theorems which have an appropriate position in this Chapter, and which will be useful in the sequel. The Calculus is also known by the name of "The method of separation of symbols of operation from symbols of quantity;" and first I must premise a few remarks on its principles.

Let a and b be symbols of operations, and u and v be subjects on which they are performed; then, if

$$a.b(u) = b.a(u), \tag{1}$$

that is, if the order in which the operations are performed may be interchanged without an alteration of result, the operations are said to be *commutative*, and the law expressed in (1) is called the *commutative law*.

Again, suppose that a, u, v are such that

$$a(u+v) = a(u) + a(v), \tag{2}$$

the operation is said to be *distributive*, and the law is called the *distributive law*.

These laws are manifestly satisfied in the case of constants; for suppose c and c' to be constants, then

$$cc'u = c'cu,$$

$$c(u+v) = cu + cv,$$

and are not satisfied in the case of $a = \log_e$, or $b = \sin$, for $\log_e \sin u$ is not equal to $\sin \log_e u$; neither is $\sin(u+v)$ equal to $\sin u + \sin v$.

Another law which is sometimes cited as a fundamental law of the Calculus of operations is

$$a^m\, a^n (u) = a^{m+n} (u), \qquad (3)$$

and is by Mr. Gregory* called the *law of repetition;* but it is not so much a law of combination of symbols of operation as an assertion that the symbol a is subject to the index law, and that the operation expressed by it is performed $m+n$ times, m times, that is, on the back of the operation performed n times previously; it is in fact rather a law of notation than a law of combination.

It is the general results of these laws, in the particular case of Differential Calculus, that we are about to trace: our object is to infer certain truths contained in them, and by means of them to generalize several previous equations.

To give a notion of the process, let us consider the Binomial Theorem of common algebra. By multiplying together factors of the form $x-a$, $x-b$, ... we are led to a general form which is applicable when n such multipliers are multiplied, and by making $a = b = c$......, we are led to a general value of $(x-a)^n$; this acts a suggestive part as to the law of combination of such factors, and from our knowledge that such a result not only is true but must be true, we infer *deductively* that the formula holds good when n is general in form as well as in value; that is, the Binomial Theorem is true for fractional and negative as well as for integral and positive indices. It is true also when a and b are symbols of commutative and distributive operations, for no other conditions as to a and b are required except that they satisfy these two laws; hence an examination of the particular exhibits such a law of combination that it must be true in the general case; and the result is no case of induction, for the truth of the general proposition is not inferred from that of the particular, but follows directly by necessity of the law; and that such is the case, is shewn by Euler's proof of the Binomial Theorem.

365.] Now let us examine how far the results of differentiation admit of such an extension. In the first place the

* See Gregory's Examples, Chap. XV.

commutative law is satisfied; for if u be a function of two variables x and y, by Art. 73

$$\frac{d}{dy}\cdot\frac{d}{dx}(u) = \frac{d}{dx}\cdot\frac{d}{dy}(u). \tag{4}$$

The distributive law is also satisfied; for if u and v be functions of x,

$$\frac{d}{dx}(u+v) = \frac{d}{dx}(u) + \frac{d}{dx}(v), \tag{5}$$

and

$$\frac{d^n}{dx^n}(u+v) = \frac{d^n u}{dx^n} + \frac{d^n v}{dx^n};$$

and the notation is also in accordance with the index law, for

$$\frac{d^m}{dx^m}\,\frac{d^n}{dx^n}(u) = \frac{d^{m+n}}{dx^{m+n}}(u). \tag{6}$$

And if $u = F(x, y, z \ldots\ldots)$, by equation (90) Art. 73 we have

$$\frac{d^{r+s+t+\ldots}u}{dx^r\,dy^s\,dz^t\ldots} = \frac{d^{s+t+r+\ldots}u}{dy^s\,dz^t\,dx^r\ldots} = \ldots\ldots$$

which is in accordance with both the commutative and index laws.

In what follows I shall not consider the meaning of differential coefficients when the indices such as m and n in (6) are fractional, for the theory is too imperfect to find a place in an elementary treatise, though some advance has lately been made in it; and I must also anticipate the notation of the Integral Calculus, so far as to replace $\left(\frac{d}{dx}\right)^{-1}$ by $\int dx$, whereby

$$\frac{d^{-n}}{dx^{-n}} = \int^n dx^n, \tag{7}$$

$$\int^n dx^n\,\frac{d^m}{dx^m}(u) = \frac{d^{m-n}}{dx^{m-n}}(u). \tag{8}$$

366.] If u be a function of two independent variables x and y, then by Art. 47

$$\mathrm{D}u = \left(\frac{du}{dx}\right)dx + \left(\frac{du}{dy}\right)dy; \tag{9}$$

whence, separating the symbols of operation from their subjects,

$$\text{D} = \left(\frac{d}{dx}\right) dx + \left(\frac{d}{dy}\right) dy; \tag{10}$$

that is, the operation expressed by D is equivalent to that expressed by the right-hand member of (10).

Let the operation D be performed n times successively on u, in which case x and y must increase by the same increments as at first, or be equicrescent, then

$$\text{D}^n = \left\{\left(\frac{d}{dx}\right) dx + \left(\frac{d}{dy}\right) dy\right\}^n; \tag{11}$$

and therefore expanding by the Binomial Theorem,

$$\text{D}^n = \left(\frac{d^n}{dx^n}\right) dx^n + \frac{n}{1}\left(\frac{d^n}{dx^{n-1}dy}\right) dx^{n-1} dy + \dots\dots \tag{12}$$

and adding the subject

$$\text{D}^n u = \left(\frac{d^n u}{dx^n}\right) dx^n + \frac{n}{1}\left(\frac{d^n u}{dx^{n-1}dy}\right) dx^{n-1} dy + \dots\dots \tag{13}$$

the same result as that of Art. 74, equation (104).

And employing the notation of Art. 47,

$$\begin{aligned} \text{D}u &= d_x u + d_y u, \\ \text{D} &= d_x + d_y, \\ \therefore\quad \text{D}^n &= (d_x + d_y)^n, \\ \text{D}^n u &= (d_x + d_y)^n u, \\ &= d_x{}^n u + n\, d_x{}^{n-1} d_y u + \frac{n(n-1)}{1.2} d_x{}^{n-2} d_y^2 u + \dots\dots \end{aligned}$$

Similarly if u be a function of three or more independent variables, since by Art. 49

$$\text{D}u = \left(\frac{du}{dx}\right) dx + \left(\frac{du}{dy}\right) dy + \left(\frac{du}{dz}\right) dz + \dots\dots$$

$$\therefore\quad \text{D} = \left(\frac{d}{dx}\right) dx + \left(\frac{d}{dy}\right) dy + \left(\frac{d}{dz}\right) dz + \dots\dots$$

$$\text{D}^n = \left\{\left(\frac{d}{dx}\right) dx + \left(\frac{d}{dy}\right) dy + \left(\frac{d}{dz}\right) dz + \dots\dots\right\}^n \tag{14}$$

and expanding the right-hand member by the Multinomial

Theorem, and appending the subject u, we obtain a series of terms expressed in the form

$$\text{D}^n u = 1.2.3\ldots n \sum \left(\frac{d^n u}{dx^\alpha\, dy^\beta\, dz^\gamma\ldots}\right) \frac{dx^\alpha\, dy^\beta\, dz^\gamma\ldots}{1.2\ldots\alpha 1.2\ldots\beta 1.2\ldots\gamma\ldots} \tag{15}$$

where $\qquad n = \alpha + \beta + \gamma + \ldots\ldots$

Thus, if

$$u = e^{ax+by+cz+\ldots},$$

$$\text{D}u = e^{ax+by+cz+\ldots}\{a\,dx + b\,dy + c\,dz + \ldots\ldots\},$$

$$\text{D}^n u = e^{ax+by+cz+\ldots}\{a\,dx + b\,dy + c\,dz + \ldots\ldots\}^n.$$

367.] Let u and v be two functions of x, then, by Art. 28,

$$\frac{d.uv}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}, \tag{16}$$

which may be written in the form

$$\frac{d}{dx}uv = \frac{d.uv}{dx} + \frac{d'.uv}{dx}, \tag{17}$$

if d (unaccentuated) refers to u exclusive of v, and if d' refers to v exclusive of u; and taking therefore symbols of operation alone, we have

$$\frac{d}{dx} = \frac{d}{dx} + \frac{d'}{dx}, \tag{18}$$

$$\therefore \quad \frac{d^n}{dx^n} = \left(\frac{d}{dx} + \frac{d'}{dx}\right)^n;$$

and expanding the Binomial expression on the right-hand side,

$$\frac{d^n}{dx^n} = \frac{d^n}{dx^n} + n\frac{d^{n-1}d'}{dx^n} + \frac{n(n-1)}{1.2}\frac{d^{n-2}d'^2}{dx^n} + \ldots\ldots \tag{19}$$

and appending the subject uv,

$$\frac{d^n.uv}{dx^n} = v\frac{d^n u}{dx^n} + n\frac{d^{n-1}u}{dx^{n-1}}\frac{dv}{dx} + \frac{n(n-1)}{1.2}\frac{d^{n-2}u}{dx^{n-2}}\frac{d^2v}{dx^2} + \ldots \tag{20}$$

which is the theorem of Leibnitz, proved before in Art. 53.

Let n be negative, then, by reason of equation (7),

$$\int^n uv\,dx^n = v\int^n u\,dx^n - n\frac{dv}{dx}\int^{n+1} u\,dx^{n+1}$$

$$+ \frac{n(n+1)}{1.2}\frac{d^2v}{dx^2}\int^{n+2} u\,dx^{n+2} + \ldots\ldots \tag{21}$$

In this last expression let $u = 1$, $n = 1$,

$$\int v\,dx = v\int dx - \frac{dv}{dx}\int^2 dx^2 + \frac{d^2v}{dx^2}\int^3 dx^3 + \ldots\ldots \tag{22}$$

In (20) let $v = e^{ax}$, then

$$\frac{d^n.ue^{ax}}{dx^n} = e^{ax}\frac{d^n u}{dx^n} + na\,e^{ax}\frac{d^{n-1}u}{dx^{n-1}} + \frac{n(n-1)}{1.2}a^2e^{ax}\frac{d^{n-2}u}{dx^{n-2}} + \ldots$$

$$= e^{ax}\left\{\frac{d^n}{dx^n} + na\frac{d^{n-1}}{dx^{n-1}} + \frac{n(n-1)a^2}{1.2}\frac{d^{n-2}}{dx^{n-2}} + \ldots\ldots\right\}u; \tag{23}$$

$$\therefore\ \left(\frac{d}{dx}\right)^n ue^{ax} = e^{ax}\left(\frac{d}{dx} + a\right)^n u,$$

and therefore

$$\left(\frac{d}{dx} + a\right)^n u = e^{-ax}\left(\frac{d}{dx}\right)^n e^{ax}u. \tag{24}$$

368.] Again by Taylor's Series, equation (76), Art. 66, replacing $f'(x)$, $f''(x)$, ... in terms of differential coefficients,

$$f(x+h) = f(x) + \frac{d.f(x)}{dx}\frac{h}{1} + \frac{d^2.f(x)}{dx^2}\frac{h^2}{1.2} + \frac{d^3.f(x)}{dx^3}\frac{h^3}{1.2.3} + \ldots\ldots \tag{25}$$

therefore

$$f(x+h) = \left\{1 + \frac{d}{dx}\frac{h}{1} + \frac{d^2}{dx^2}\frac{h^2}{1.2} + \frac{d^3}{dx^3}\frac{h^3}{1.2.3} + \ldots\right\}f(x), \tag{26}$$

$$= e^{h\frac{d}{dx}}f(x), \tag{27}$$

as is manifest from the form of the exponential theorem; that is, $f(x)$ being operated upon by a series of operations, the sum of which is expressed by $e^{h\frac{d}{dx}}$, is converted into $f(x+h)$.

For convenience of notation let us, after M. Servois, represent $e^{\frac{d}{dx}}$ by E; therefore

$$f(x+h) = \text{E}^h f(x). \tag{28}$$

Hence $$f(x+h+k) = \text{E}^k\,\text{E}^h f(x),$$

$$= \text{E}^{h+k} f(x), \tag{29}$$

and therefore generally,

$$f(x+h+k+l+\ldots\ldots) = \text{E}^{h+k+l+\ldots} f(x).$$

We may employ the above symbolical form of Taylor's Series in the solution of problems such as those given in Ex. 4 and 5, Art. 68. For suppose that $f(x)\,f(h) = f(x+h)$, and that the problem is to determine the form of the function, then since

$$f(x+h) = \text{E}^h f(x),$$

$$\therefore\quad f(x)\,f(h) = \text{E}^h f(x),$$

$$f(h) = \text{E}^h;$$

and as E is the symbol of $e^{\frac{d}{dx}}$, which is independent of h, we may consider it constant in reference to h, and write a for E, so that

$$f(h) = a^h,$$

$$\therefore\quad f(x) = a^x,$$

$$f(x)\,f(h) = a^x \times a^h = a^{x+h},$$

$$= f(x+h);$$

similarly may other problems of the same kind be solved.

369.] Again, suppose that f is the symbol of a function or series of functions which satisfy the *commutative* and *distributive* laws, and suppose also π to be a general symbol of operation, then, in the employment of such a function of operations as $f(\pi)$, it is subject to the condition that $f(\pi)$ is susceptible of being developed in terms of functions which also satisfy both these laws; that is, $f(\pi)$ when expanded consists of a series of terms of the form π^m, and, when the subject-symbol u is appended, $\pi^m(u)$ indicates that the operation symbolized by π is performed on u m times successively. Thus the conditions are satisfied if $f(\pi)$ admits of being expanded in the form of Maclaurin's Series.

Suppose that f is a symbol of this nature; then, since

$$\frac{d}{dx}\left\{u e^{\phi(x)}\right\} = e^{\phi(x)}\left\{\frac{du}{dx} + u\,\phi'(x)\right\},$$

$$= e^{\phi(x)}\left\{\frac{d}{dx} + \phi'(x)\right\} u,$$

$$\therefore \quad \left\{\frac{d}{dx} + \phi'(x)\right\} u = e^{-\phi(x)}\frac{d}{dx}\left\{u e^{\phi(x)}\right\}; \tag{30}$$

hence f being such as we suppose,

$$f\left\{\frac{d}{dx} + \phi'(x)\right\} u = e^{-\phi(x)} f\left\{\frac{d}{dx}\right\} e^{\phi(x)} u. \tag{31}$$

In the particular case in which

$$f\left\{\frac{d}{dx} + \phi'(x)\right\} = \left\{\frac{d}{dx} + \phi'(x)\right\}^n,$$

$$\left\{\frac{d}{dx} + \phi'(x)\right\}^n u = e^{-\phi(x)}\left(\frac{d}{dx}\right)^n e^{\phi(x)} u; \tag{32}$$

and if $e^{\phi(x)} = e^{ax}$,

$$\left\{\frac{d}{dx} + a\right\}^n u = e^{-ax}\left(\frac{d}{dx}\right)^n e^{ax} u; \tag{33}$$

and if $n = -1$,

$$\left\{\frac{d}{dx} + a\right\}^{-1} u = e^{-ax}\left(\frac{d}{dx}\right)^{-1} e^{ax} u,$$

$$= e^{-ax}\int e^{ax} u\, dx. \tag{34}$$

Thus equation (24) is only a particular case of the more general theorem contained in (31).

370.] Again, in (31), let $e^{\phi(x)} = e^{ax}$,

$$\therefore \quad e^{ax} f\left\{\frac{d}{dx} + a\right\} u = f\left\{\frac{d}{dx}\right\} e^{ax} u. \tag{35}$$

For e^x write x, and therefore for dx write $\frac{dx}{x}$; then

$$x^a f\left\{x\frac{d}{dx} + a\right\} u = f\left\{x\frac{d}{dx}\right\} x^a u. \tag{36}$$

The above is the merest outline of the principles of the

Calculus of operations, and of some of its most simple results; but as they are sufficient for the use which will be made of it in the sequel, it is unnecessary to add more: especially too as many parts of it are but ill-understood at present, and as a full discussion of these questions requires more space than we can afford to give to it in the present volume. To the student however who desires to enter more deeply into the subject, I would recommend the study of (1) many papers by Mr. D. F. Gregory, Mr. G. Boole, Mr. Donkin, Mr. Bronwin, and others which are contained in the volumes of the Cambridge and Dublin Mathematical Journal; (2) the able memoir of Mr. G. Boole, entitled "On a General Method of Analysis," and inserted in the Philosophical Transactions, Part II, 1844, and which was most deservedly rewarded with a medal; (3) the papers of M. Servois in the Annales des Mathématiques, Vol. V; (4) Gregory's Examples on the Differential Calculus, Chapter XV. Probably also he will find some valuable information on the same subject in "Essai sur un nouveau mode d'exposition des Principes de Calcul Différentiel," by M. Servois, Bachelier, Paris; but I have been unable to procure the work.

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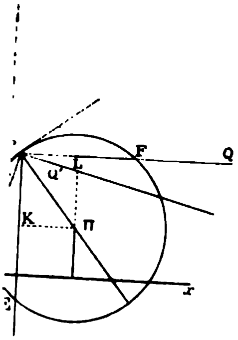

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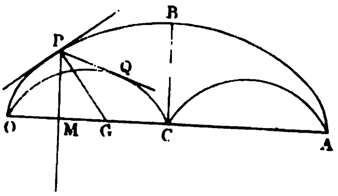

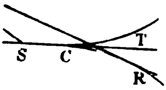

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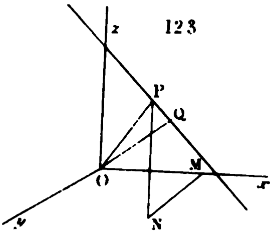

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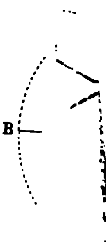

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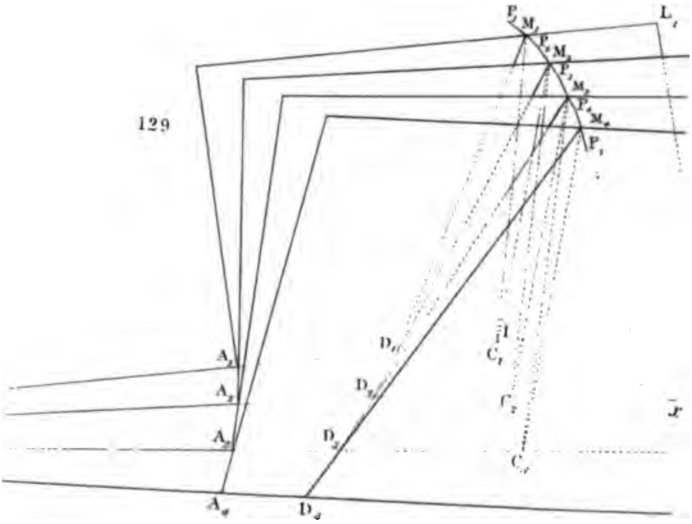

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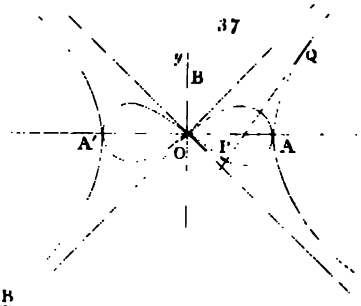

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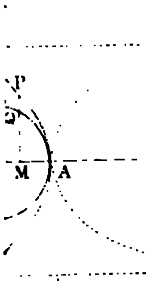

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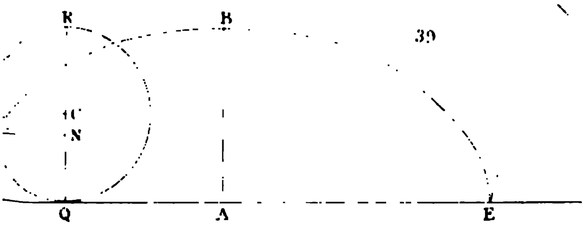

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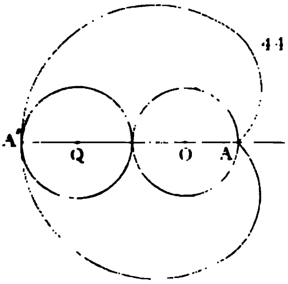

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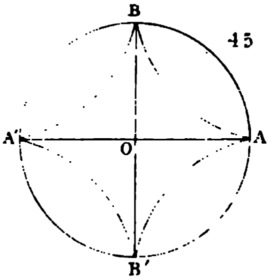

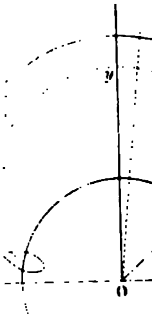

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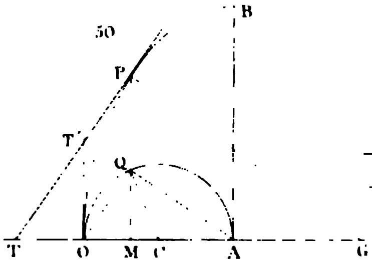

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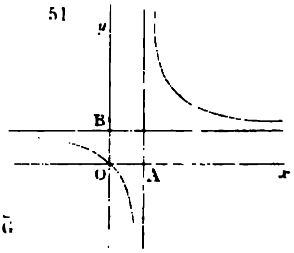

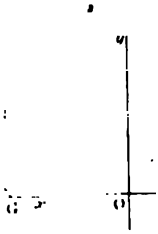

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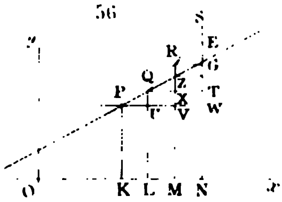

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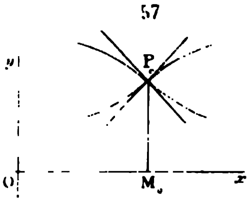

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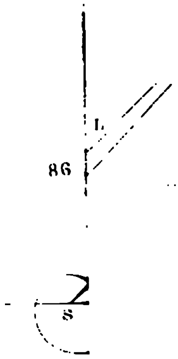

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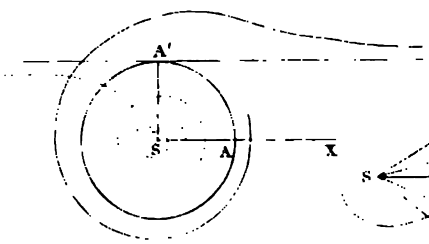

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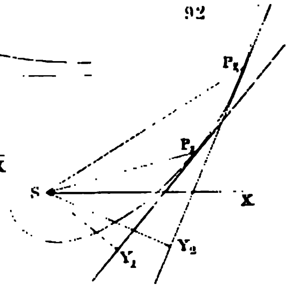

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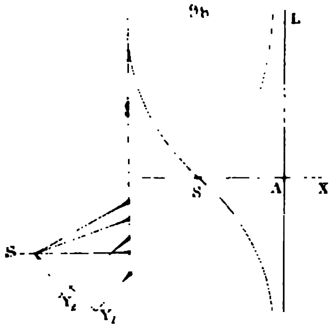

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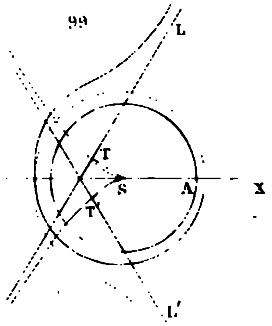

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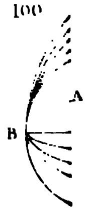

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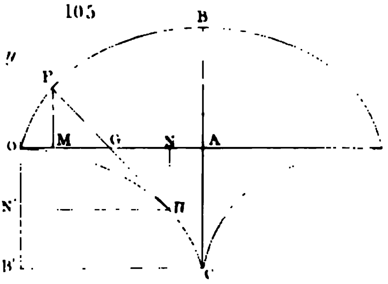

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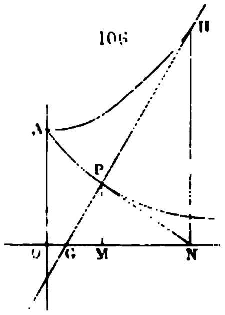

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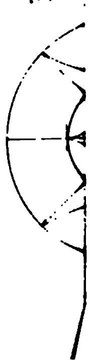

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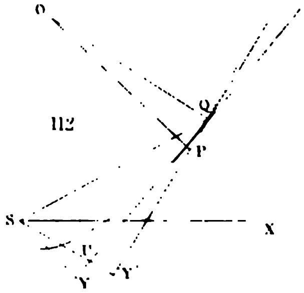

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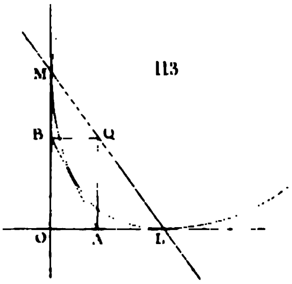

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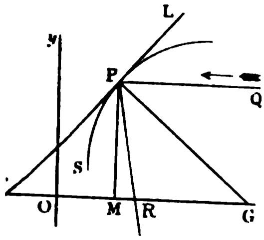

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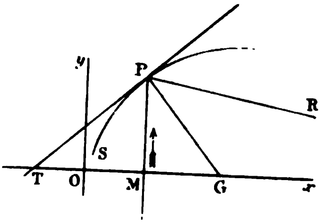

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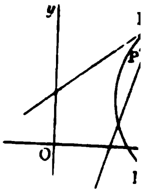

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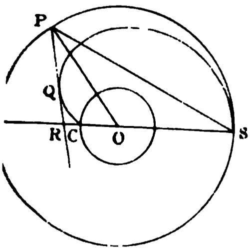

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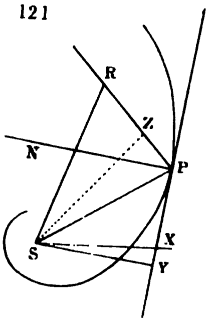

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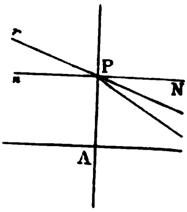

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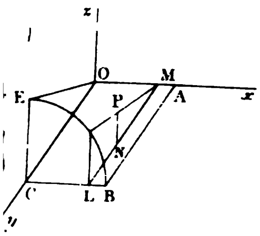

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